Jan 16, 2004.

2.1 Constructing Difference Operators.

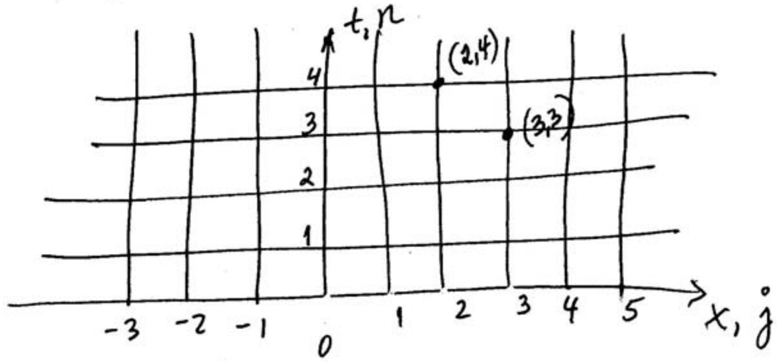
We will start constructing finite difference approxs. or difference schemes to 1-D problems as

1) $\begin{cases} u_t + a u_x = 0, & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) & \text{Simple wave equation} \end{cases}$

2) $\begin{cases} u_t = \sigma u_{xx}, & 0 < x < 1, \quad t > 0 \\ u(x,0) = \phi(x), & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & \text{Heat conduction problem.} \end{cases}$

Obviously, we need finite difference approximations for the derivative terms involved in these equations.

For example the domain for problem ① is the upper half plane.



We make a partition as indicated:

Uniform partition of cells of size $\Delta x \times \Delta t$

These cells are obtained by making a uniform partition of the x-axis

$$x_j = j\Delta x, \quad j = \cdots -3,-2,-1,0,1,2,3,\ldots$$

and the t-axis: $t_n = n\Delta t, \quad n = 0,1,2,\ldots$

The intersection of the vertical line: $x = x_j$

and the horizontal line: $t = t_n$

is the grid point: (j,n).

And the value of the solution $u(x,t)$ at the grid point: (x_j, t_n) or (j,n) is denoted as

$$u(x_j,t_n) = u(j\Delta x, n\Delta t) = u_j^n$$

The finite difference approx. of u_j^n is U_j^n.

Construction of Finite Differences approximation

Keystone: Taylor's Thm.

If $f'(x), \ldots, f^{(k+1)}(x)$ are defined on $[a, x]$ then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\eta)}{(k+1)!}(x-a)^{k+1},$$

$\eta \in (a, x)$.

In particular, if $f(x) = u(x, t_n)$, for t_n fixed and $f(a) = u(x_j, t_n)$ and $f(x) = u(x_{j+1}, t_n)$, then $\underset{\Delta x}{\overset{x-a}{\shortparallel}} = x_{j+1} - x_j$

and

$$u(x_{j+1}, t_n) = u_{j+1}^n = u_j^n + \left(\frac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2} + \cdots$$

$$+ \cdots + \frac{1}{k!}\left(\frac{\partial^k u}{\partial x^k}\right)_j^n \Delta x^k + \underbrace{\frac{1}{(k+1)!}\left(\frac{\partial^{k+1} u}{\partial x^{k+1}}(\eta, t_n)\right)\Delta x^{k+1}}_{\text{remainder}}$$

Since $\eta \in (x_j, x_{j+1}) \Rightarrow \eta = (j+\xi)\Delta x$ with $\xi \in (0,1)$.

remainder.

If we only consider two terms and the remainder

$$\boxed{u_{j+1}^n = u_j^n + \left(\frac{\partial u}{\partial x}\right)_j^n \Delta x + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{j+\xi}^n \Delta x^2 .}$$

Now, solving for $\left(\frac{\partial u}{\partial x}\right)_j^n$

$$(u_x)_j^n = \left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} \underbrace{- \frac{1}{2}(u_{xx})_{j+\xi}^n \Delta x}_{\text{remainder}} .$$

Neglecting the remainder

$$\frac{U_{j+1}^n - U_j^n}{\Delta x} = (U_x)_j^n . \qquad (4.1)$$

Called first forward finite difference approximation.

Also,

$$\tau_j^n = \text{local truncation error} = -\frac{1}{2}(u_{xx})_{j+\xi}^n \Delta x , \qquad 0<\xi<1 .$$

and we say that (4.1) forward diffce approx. is $\mathcal{O}(\Delta x)$. "of order Δx".

Similarly, we can obtain the formula for BACKWARD FINITE DIFFERENCE

$$(U_x)_j^n = \frac{U_j^n - U_{j-1}^n}{\Delta x} \qquad (5.1)$$

With local discretization error:

$$\tau_j^n = \frac{1}{2} (u_{xx})_{j-\alpha}^n , \qquad 0 < \alpha < 1$$

CENTERED FINITE DIFFERENCE

$$u_{j+1}^n = \cancel{u_j^n} + (u_x)_j^n \Delta x + \cancel{\frac{1}{2} (u_{xx})_j^n \Delta x^2} + \frac{1}{3!} (u_{xxx})_{j+\xi}^n \Delta x^3 \qquad (5.2)$$

$$-\left[u_{j-1}^n = \cancel{u_j^n} - (u_x)_j^n \Delta x + \cancel{\frac{1}{2} (u_{xx})_j^n \Delta x^2} - \frac{1}{3!} (u_{xxx})_{j-\alpha} \Delta x^3 \right] \qquad (5.3)$$

$$u_{j+1}^n - u_{j-1}^n = 2 (u_x)_j^n \Delta x + \frac{1}{3!} \left[(u_{xxx})_{j+\xi}^n + (u_{xxx})_{j-\alpha}^n \right] \Delta x^3$$

$$0 < \xi < 1, \quad 0 < \alpha < 1.$$

Thus, solving for $(u_x)_j^n$

$$(u_x)_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x} - \frac{1}{6} \left[\frac{(u_{xxx})_{j+\xi}^n + (u_{xxx})_{j-\alpha}^n}{2} \right] \Delta x^2$$

The local truncation error

$$\tau_j^n = -\frac{1}{6}\left[\frac{(u_{xxx})_{j+\xi}^n + (u_{xxx})_{j-\alpha}^n}{2}\right]\Delta x^2,$$

using the intermediate value thm for conts. function can be written as

$$\boxed{\tau_j^n = -\frac{1}{6}\,(u_{xxx})_{j+\beta}^n\,\Delta x^2;} \quad (6.1) \quad -1<\beta<1.$$

Why?.

Therefore, the centered finite diffce approximation of the first derivative $(u_x)_j^n$ is given by

$$(U_x)_j^n = \frac{U_{j+1}^n - U_{j-1}^n}{2\,\Delta x}$$

With local truncation error (6.1). We can see that the advantage of this formula is that it is of order Δx^2, or $\mathcal{O}(\Delta x^2)$.

Exactly the same procedure can be applied for finite diffce approximations for the time derivative $(u_t)_j^n$. In fact,

$$(u_t)_j^n \simeq (U_t)_j^n = \frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t},$$

Centered finite difference.

Second order centered finite difference approximation of $(U_{xx})_j^n$ can be obtained by retaining one more term in (5.2) and (5.3) and adding the resulting expressions

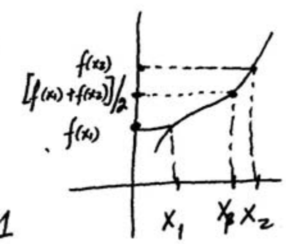
Do it!

$$\left(U_{xx}\right)_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

with truncation error

$$\tau_j^n = -\frac{1}{12}\left[\frac{(U_{xxxx})_{j+\xi}^n + (U_{xxxx})_{j-\alpha}^n}{2}\right]\Delta x^2$$

$$\overset{\text{Int. Value Thm}}{=} -\frac{1}{12}\left(U_{xxxx}\right)_{j+\beta}^n, \qquad -1<\beta<1$$



$x_1 < x_\beta < x_2$

Non-Uniform Grids:

$$\overbrace{x_{j-1} \quad\quad}^{\Delta x_L} \overbrace{x_j \quad\quad\quad\quad x_{j+1}}^{\Delta x_R}$$

Centered differences:

$$\left(U_x\right)_j^n = \frac{1}{2}\left[\frac{U_{j+1}^n - U_j^n}{\Delta x_R} + \frac{U_j^n - U_{j-1}^n}{\Delta x_L}\right] - \frac{1}{4}\left(\Delta x_R - \Delta x_L\right)\left(U_{xx}\right)_j^n + \mathcal{O}\left(\max(\Delta x_L^2, \Delta x_R^2)\right)$$

Clearly, if $\Delta x_R \neq \Delta x_L$ formula is only $\mathcal{O}(\Delta x)$ where $\Delta x = \max(\Delta x_L, \Delta x_R)$.

Example: Obtain the one-side three points formula:

$$(u_x)_j = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x_j} + \mathcal{O}(\Delta x^2).$$

<u>Derivation</u>: Find a, b, and c that satisfies the equation.

$$(u_x)_j = a\,u_{j+2} + b\,u_{j+1} + c\,u_j + \mathcal{O}(\Delta x^2).$$

$$a\,u_{j+2} = a\left[u_j + 2\Delta x\,(u_x)_j + \frac{4\Delta x^2}{2}(u_{xx})_j + \frac{8\Delta x^3}{3!}(u_{xxx})_{j+\xi}\right.$$

$$b\,u_{j+1} = b\left[u_j + \Delta x\,(u_x)_j + \frac{\Delta x^2}{2}(u_{xx})_j + \frac{(\Delta x)^3}{3!}(u_{xxx})_{j+\eta}\right.$$

$$c\,u_j = c\,u_j$$

$$\Rightarrow a\,u_{j+2} + b\,u_{j+1} + c\,u_j = (a+b+c)u_j + (2a+b)(u_x)_j\,\Delta x +$$

$$+\left(2a + \frac{b}{2}\right)(u_{xx})_j + \mathcal{O}(\Delta x^3).$$

If
(1) $a+b+c = 0$
(2) $2a+b = 1$
(3) $2a + \frac{b}{2} = 0$

$\Rightarrow (u_x)_j\,\Delta x = \frac{1}{\Delta x}\left[a\,u_{j+2} + b\,u_{j+1} + c\,u_j\right] + \mathcal{O}(\Delta x^2)$

(2) − (3) $\Rightarrow \frac{b}{2} = 1 \Rightarrow \boxed{b=2}$

$\Rightarrow \boxed{a = -\frac{1}{2}} \Rightarrow c = -a-b = -\frac{3}{2}$

$\boxed{c = -3/2}$

$$\Rightarrow (u_x)_j = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x_j} + \mathcal{O}(\Delta x^2)$$

✓

$\therefore$

$$(u_x)_j = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x} + O(\Delta x^2).$$

Homework: Show that

① $$(u_x)_j = \frac{1}{12}\left[u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}\right] + O(\Delta x^4).$$

5 points formula.

② Obtain from Table 3.1 Tannehill.

a) (3.42), b) (3.52).

Table 3.1. Comparison of formulae to evaluate $d\bar{T}/dx$ at $x=1.0$ for $\bar{T}(x) = e^x$

Case	Algebraic formula	$\left[\frac{d\bar{T}}{dx}\right]_j$	Error	Leading term in T.E.
Exact	—	2.7183	—	—
3PT SYM	$(\bar{T}_{j+1}-\bar{T}_{j-1})/2\Delta x$	2.7228	0.4533×10^{-2}	0.4531×10^{-2}
FOR DIFF	$(\bar{T}_{j+1}-\bar{T}_j)/\Delta x$	2.8588	0.1406×10^{-0}	0.1359×10^{-0}
BACK DIFF	$(\bar{T}_j-\bar{T}_{j-1})/\Delta x$	2.5868	-0.1315×10^{-0}	-0.1359×10^{-0}
3PT ASYM	$(-1.5\bar{T}_j+2\bar{T}_{j+1}-0.5\bar{T}_{j+2})/\Delta x$	2.7085	-0.9773×10^{-2}	-0.9061×10^{-2}
5PT SYM	$(\bar{T}_{j-2}-8\bar{T}_{j-1}+8\bar{T}_{j+1}-\bar{T}_{j+2})/12\Delta x$	2.7183	-0.9072×10^{-5}	-0.9061×10^{-5}

Table 3.2. Comparison of formulae to evaluate $d^2\bar{T}/dx^2$ at $x=1.0$

Case	Algebraic formula	$\left[\frac{d\bar{T}^2}{dx^2}\right]_j$	Error	Leading term in T.E.
Exact	—	2.7183	—	—
3PT SYM	$(\bar{T}_{j-1}-2\bar{T}_j+\bar{T}_{j+1})/\Delta x^2$	2.7205	0.2266×10^{-2}	0.2265×10^{-2}
3PT ASYM	$(\bar{T}_j-2\bar{T}_{j+1}+\bar{T}_{j+2})/\Delta x^2$	3.0067	0.2884×10^{-0}	0.2718×10^{-0}
5PT SYM	$(-\bar{T}_{j-2}+16\bar{T}_{j-1}-30\bar{T}_j+16\bar{T}_{j+1}-\bar{T}_{j+2})/12\Delta x^2$	2.7183	-0.3023×10^{-5}	-0.3020×10^{-5}

Table 3.3. Truncation error leading term (algebraic): $d\bar{T}/dx$

Case	Algebraic formula	Truncation error leading term
3PT SYM	$(\bar{T}_{j+1}-\bar{T}_{j-1})/2\Delta x$	$\Delta x^2\bar{T}_{xxx}/6$
FOR DIFF	$(\bar{T}_{j+1}-\bar{T}_j)/\Delta x$	$\Delta x\bar{T}_{xx}/2$
BACK DIFF	$(\bar{T}_j-\bar{T}_{j-1})/\Delta x$	$-\Delta x\bar{T}_{xx}/2$
3PT ASYM	$(-1.5\bar{T}_j+2\bar{T}_{j+1}-0.5\bar{T}_{j+2})/\Delta x$	$-\Delta x^2\bar{T}_{xxx}/3$
5PT SYM	$(\bar{T}_{j-2}-8\bar{T}_{j-1}+8\bar{T}_{j+1}-\bar{T}_{j+2})/12\Delta x$	$-\Delta x^4\bar{T}_{xxxxx}/30$

Table 3.4. Truncation error leading term (algebraic): $d^2\bar{T}/dx^2$

Case	Algebraic formula	Truncation error leading term
3PT SYM	$(\bar{T}_{j-1}-2\bar{T}_j+\bar{T}_{j+1})/\Delta x^2$	$\Delta x^2\bar{T}_{xxxx}/12$
3PT ASYM	$(\bar{T}_j-2\bar{T}_{j+1}+T_{j+2})/\Delta x^2$	$\Delta x\bar{T}_{xxx}$
5PT SYM	$(-\bar{T}_{j-2}+16\bar{T}_{j-1}-30\bar{T}_j+16\bar{T}_{j+1}-\bar{T}_{j+2})/12\Delta x^2$	$-\Delta x^4\bar{T}_{xxxxxx}/90$

Table 3-1 Difference approximations using more than three points

Derivative	Finite-difference representation	Equation
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} + O(h^2)$	(3-38)
$\left.\dfrac{\partial^4 u}{\partial x^4}\right)_{i,j} =$	$\dfrac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} + O(h^2)$	(3-39)
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{-u_{i+3,j} + 4u_{i+2,j} - 5u_{i+1,j} + 2u_{i,j}}{h^2} + O(h^2)$	(3-40)
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{-3u_{i+4,j} + 14u_{i+3,j} - 24u_{i+2,j} + 18u_{i+1,j} - 5u_{i,j}}{2h^3} + O(h^2)$	(3-41)
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{2u_{i,j} - 5u_{i-1,j} + 4u_{i-2,j} - u_{i-3,j}}{h^2} + O(h^2)$	(3-42)
$\left.\dfrac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\dfrac{5u_{i,j} - 18u_{i-1,j} + 24u_{i-2,j} - 14u_{i-3,j} + 3u_{i-4,j}}{2h^3} + O(h^2)$	(3-43)
$\left.\dfrac{\partial u}{\partial x}\right)_{i,j} =$	$\dfrac{-u_{i+2,j} + 8u_{i+1,j} - 8u_{i-1,j} + u_{i-2,j}}{12h} + O(h^4)$	(3-44)
$\left.\dfrac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\dfrac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12h^2} + O(h^4)$	(3-45)

Table 3-2 Difference approximations for mixed partial derivatives

Derivative	Finite-difference representation	Equation
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \dfrac{u_{i,j} - u_{i,j-1}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-46)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j+1} - u_{i,j}}{\Delta y} - \dfrac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-47)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j} - u_{i,j-1}}{\Delta y} - \dfrac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-48)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \dfrac{u_{i,j+1} - u_{i,j}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-49)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta y} - \dfrac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y}\right) + O[\Delta x, (\Delta y)^2]$	(3-50)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{\Delta x}\left(\dfrac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y} - \dfrac{u_{i-1,j+1} - u_{i-1,j-1}}{2\,\Delta y}\right) + O[\Delta x, (\Delta y)^2]$	(3-51)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta y} - \dfrac{u_{i-1,j+1} - u_{i-1,j-1}}{2\,\Delta y}\right) + O[(\Delta x)^2, (\Delta y)^2]$	(3-52)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \dfrac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y}\right) + O[(\Delta x)^2, \Delta y]$	(3-53)
$\left.\dfrac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\dfrac{1}{2\,\Delta x}\left(\dfrac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \dfrac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y}\right) + O[(\Delta x)^2, \Delta y]$	(3-54)

Hornbeck (1975)

	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}	
$2hf'(x_i) =$		-1	0	1		
$h^2f''(x_i) =$		1	-2	1		$+ \quad O(h)^2$
$2h^3f'''(x_i) =$	-1	2	0	-2	1	
$h^4f^{iv}(x_i) =$	1	-4	6	-4	1	

(a) Representations of $O(h)^2$

	f_{i-3}	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}	f_{i+3}	
$12hf'(x_i) =$		1	-8	0	8	-1		
$12h^2f''(x_i) =$		-1	16	-30	16	-1		$+ \quad O(h)^4$
$8h^3f'''(x_i) =$	1	-8	13	0	-13	8	-1	
$6h^4f^{iv}(x_i) =$	-1	12	-39	56	-39	12	-1	

(b) Representations of $O(h)^4$

Fig. 3.4 Central difference representations.

	f_i	f_{i+1}	f_{i+2}	f_{i+3}	f_{i+4}	f_{i+5}	
$2hf'(x_i) =$	-3	4	-1				
$h^2f''(x_i) =$	2	-5	4	-1			$+ \quad O(h)^2$
$2h^3f'''(x_i) =$	-5	18	-24	14	-3		
$h^4f^{iv}(x_i) =$	3	-14	26	-24	11	-2	

(a) Forward difference representations

	f_{i-5}	f_{i-4}	f_{i-3}	f_{i-2}	f_{i-1}	f_i	
$2hf'(x_i) =$				1	-4	3	
$h^2f''(x_i) =$			-1	4	-5	2	$+ \quad O(h)^2$
$2h^3f'''(x_i) =$		3	-14	24	-18	5	
$h^4f^{iv}(x_i) =$	-2	11	-24	26	-14	3	

(b) Backward difference representations

Fig. 3.3 Forward and backward difference representations of $O(h)^2$.

OPERATOR NOTATION: FINITE DIFFERENCE OPERATORS.

Assuming $u(x)$

Forward Diffce	Δ	$\Delta u_j = u_{j+1} - u_j$
Backward Diffce	∇	$\nabla u_j = u_j - u_{j-1}$
Central Diffce	δ	$\delta u_j = u_{j+1/2} - u_{j-1/2}$
Average	μ	$\mu u_j = (u_{j+1/2} + u_{j-1/2})/2$
Shift	E	$E u_j = u_{j+1}$
Derivative	D	$D u_j = (u_x)_j$

Example: Centered difference formula in terms of difference operators.

$$\frac{(\mu\delta)(u_j)}{\Delta x} = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

In fact,

$$\frac{1}{\Delta x}\,\mu(\delta(u_j)) = \frac{1}{\Delta x}\,\mu\left(u_{j+1/2} - u_{j-1/2}\right) =$$

$$= \frac{1}{\Delta x}\left[\mu(u_{j+1/2}) - \mu(u_{j-1/2})\right] =$$

$$= \frac{1}{\Delta x}\left[\frac{u_{j+1}}{2} + \cancel{\frac{u_j}{2}} - \cancel{\frac{u_j}{2}} - \frac{u_{j-1}}{2}\right] = \frac{u_{j+1} - u_{j-1}}{2\Delta x} \checkmark$$

Example:
$$\frac{\delta^2(u_j)}{\Delta x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} \simeq (u_{xx})_j$$

In fact,
$$\delta^2(u_j) = \delta(\delta(u_j)) = \delta(u_{j+1/2} - u_{j-1/2}) =$$
$$= \delta(u_{j+1/2}) - \delta(u_{j-1/2}) = u_{j+1} - u_j - u_j + u_{j-1} =$$
$$= u_{j+1} - 2u_j + u_{j-1} \quad ✓$$

———— o ————

Also,
$$E(u_j) = u_{j+1} = e^{\Delta x D} u_j$$

In fact,
$$u_{j+1} = u_j + \Delta x (u_x)_j + \frac{\Delta x^2}{2}(u_{xx})_j + \cdots$$
$$= u_j + \Delta x D u_j + \frac{\Delta x^2}{2} D^2 u_j + \cdots$$
$$= \left(1 + \Delta x D + \frac{\Delta x^2}{2} D^2 + \cdots\right) u_j = e^{\Delta x D} u_j$$
$$\Rightarrow \quad \boxed{E = e^{\Delta x D}} \qquad (9.1)$$

and
$$\Delta u_j = u_{j+1} - u_j = E u_j - u_j = (E-1)u_j$$
$$\Rightarrow \quad \boxed{\Delta = E - 1} \Rightarrow \boxed{E = 1 + \Delta} \qquad (9.2)$$

From (9.1) $\boxed{\ln(E) = \Delta x D.}$

So

$$\ln(E) = \ln(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots, \qquad |x| < 1.$$

Defining E^{-1}, $E^{-1}(u_j) = u_{j-1}$

$$\Rightarrow \nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$$

and

$$\Delta x D = \ln(E) = -\ln(E^{-1}) = -\ln(1-\nabla) =$$

$$= -\left[-\nabla - \frac{1}{2}\nabla^2 - \frac{1}{3}\nabla^3 - \cdots\right] = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots$$

All this formal operators algebra can be proved.

<u>Application of difference operators</u>:

One of the application is to obtain formulas of higher order easily.

<u>Example</u>: One side second order finite difference approximation for $(u_x)_j$.

$$(u_x)_j \simeq \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x} \qquad (10.1)$$

$x_j \quad x_{j+1} \quad x_{j+2}$

$$T_j = O(\Delta x^2).$$

Formula (10.1) can be obtained from

$$\Delta x\, D(u_j) \simeq \ln(E)(u_j) = \ln(1+\Delta)(u_j) =$$

only 2 terms

$$\simeq \left(\Delta - \frac{1}{2}\Delta^2\right)(u_j) = u_{j+1} - u_j - \frac{1}{2}\left[\Delta(u_{j+1}) - \Delta(u_j)\right]$$

$$= u_{j+1} - u_j - \frac{1}{2}\left[\, u_{j+2} - u_{j+1} - u_{j+1} + u_j \,\right] =$$

$$= -\frac{1}{2}u_{j+2} + 2u_{j+1} - \frac{3}{2}u_j$$

Thus,

$$\boxed{(u_x)_j = D u_j = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x}.}$$