One-dimensional Kinematic Wave Equation.

Consider the Scalar linear conservation Law IVP.

$$\begin{cases} u_t + a u_x = 0, & a>0, \quad -\infty < x < \infty. \quad (1) \\ u(x,0) = \phi(x), & -\infty < x < \infty. \quad (2) \end{cases}$$

Thm.- If $\phi(x)$ is differentiable on $(-\infty,\infty)$ then, the IVP (1)-(2) has a unique Soln.

Proof.- First, we will show that

$u(x,t) = \phi(x-at)$ Satisfies (1) and (2).

Secondly, we will show that any solution of (1) and (2) needs to be

$$u(x,t) = \phi(x-at).$$

In fact, if $u(x,t) = \phi(x-at)$ then

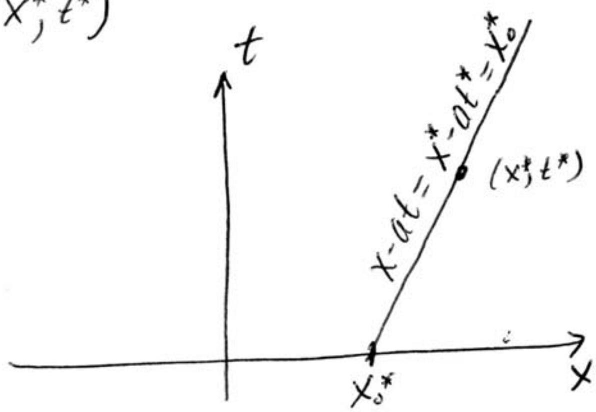
$$u_t + a u_x = \phi'(x-at)(-a) + a\,\phi'(x-at)\cdot 1 = 0 \checkmark$$

Satisfies (1).

Also, $u(x,0) = \phi(x-a0) = \phi(x) \checkmark$ Satisfies (2).

Therefore, $u(x,t) = \phi(x-at)$ is a Soln of (1)-(2).

On the other hand, Assume $u(x,t)$ is a soln of (1)-(2), and that (x^*, t^*) is an arbitrary point in the xt-$\overset{\text{semi}}{\text{plane}}$, then there is a line (called "characteristic line") passing through (x^*, t^*)



$$x - at = x_0^* = x^* - at^*$$

$$\Rightarrow \boxed{x(t) = at + x_0^*} \quad (4)$$

Clearly, at $t=0$
$x = x_0^*$.

Now, along the characteristic

$$u(x,t) = u(x(t), t) = \hat{u}(t)$$

and

$$\frac{d\hat{u}}{dt}(t) = \frac{\partial u}{\partial x}(\ )\frac{dx}{dt}(t) + \frac{\partial u}{\partial t}(x(t),t) =$$

$$= \frac{\partial u}{\partial x}(x(t),t)\, a + \frac{\partial u}{\partial t}(x(t),t) = 0$$

Since u is a soln of (1)

Thus,

$$\frac{d\hat{u}}{dt}(t) = 0$$

$$\Rightarrow \quad \hat{u}(t) \equiv K \text{ (constant)}$$

$$\Rightarrow \quad u(x,t) = u(x(t),t) = \hat{u}(t) \equiv K$$

along the characteristic passing through (x^*, t^*).

In particular, at $t=0$

$$x - at\big|_{t=0} = x_0^* \Rightarrow x = x_0^*$$

$u(x,t)$ satisfies (2).

Therefore, $u(x,0) = u(x_0^*,0) = \phi(x_0^*)$

$$\Rightarrow \qquad K = \phi(x_0^*) \qquad (4)$$

$$\Rightarrow \qquad u(x,t) = \phi(x_0^*) = \phi(x-at)$$

along the characteristic line $x - at = x_0^*$

In particular, $u(x^*, t^*) = \phi(x^* - at^*)$

Now, since (x^*, t^*) is an arbitrary point in xt-plane

$\boxed{u(x,t) = \phi(x-at),}$ for all (x,t) in the semi-plane $-\infty < x < \infty$, $t > 0$.

Solve the IVP: $u_t + 2u_x = 0$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = \begin{cases} 4x, & 0 \le x \le 1 \\ 0, & x < 0, \text{ or } x > 1 \end{cases}$$

From the theory above,

$$u(x,t) = \phi(x-2t) = \begin{cases} 4(x-2t), & 0 < x-2t < 1 \\ 0, & x-2t < 0 \text{ or } x-2t > 1 \end{cases}$$