2.2 Simple Differences Schemes for a Kinematic Wave Equation.

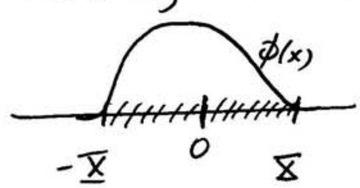
Kinematic Wave Equ.

$$\begin{cases} u_t + a(u)u_x = 0, & -\infty < x < \infty,\ t > 0 \quad (1) \\ u(x,0) = \phi(x), & -\infty < x < \infty. \quad (2) \end{cases}$$

For computational purposes, we need finite domain

This can be accomplished by asking

$$\phi(x) \equiv 0, \quad |x| > X$$



or be periodic

$$\phi(x+X) = \phi(x). \ , \quad X > 0.$$

We will use forward finite diffces to approximate the derivatives in (1)

Neglecting the local discretization errors, we obtain

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a(U_j^n)\,\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0.$$

$$v_j^{n+1} = v_j^n - \underbrace{\frac{\Delta t}{\Delta x} a(v_j^n)}_{\text{"}\alpha_j^n\text{"}} \left[v_{j+1}^n - v_j^n \right] = 0$$

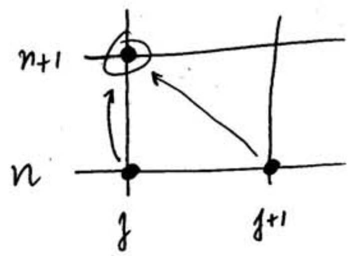
or

$$\boxed{v_j^{n+1} = (1 + \alpha_j^n) v_j^n - \alpha_j^n v_{j+1}^n} \qquad (2.1)$$

FORWARD-TIME and FORWARD-SPACE

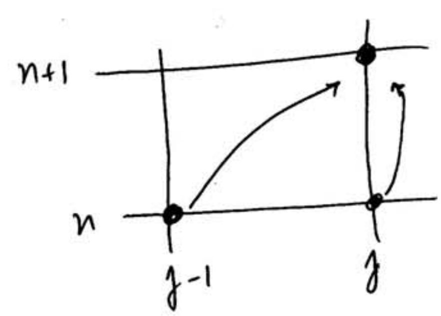
The number $\alpha_j^n = a(v_j^n) \dfrac{\Delta t}{\Delta x}$ is called the Courant number.

Formula (3.1) involves 3 points.



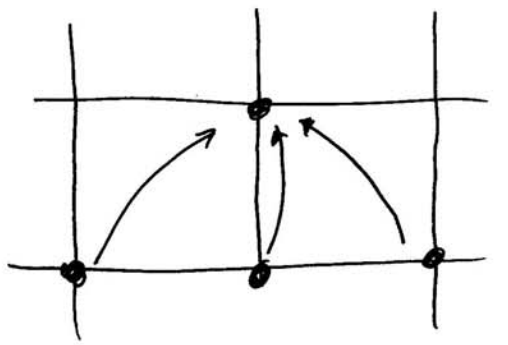
FORWARD TIME- BACKWARD SPACE

$$v_j^{n+1} = (1 - \alpha_j^n) v_j^n + \alpha_j^n v_j^{n-1} \qquad (2.2)$$



FORWARD-TIME and CENTERED-SPACE

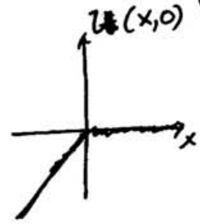
$$U_j^{n+1} = U_j^n - \frac{\alpha_j^n}{2}\left(U_{j+1}^n - U_{j-1}^n\right) \quad \text{Verify it! (3.1)}$$



Example 2.2.1 $\quad -\infty < x < \infty,\ t > 0$

$$\begin{cases} u_t + a u_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} x, & x \le 0 \\ 0, & x > 0 \end{cases} \end{cases}$$

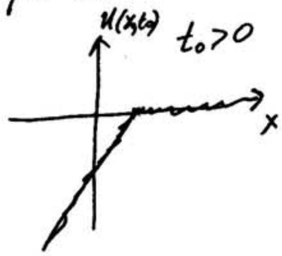
(3)



It has exact solution: $u(x,t) = \phi(x-t)$.

Obtain a numerical solution for this IVP using

$\Delta x = 1/10, \quad \Delta t = 1/20$



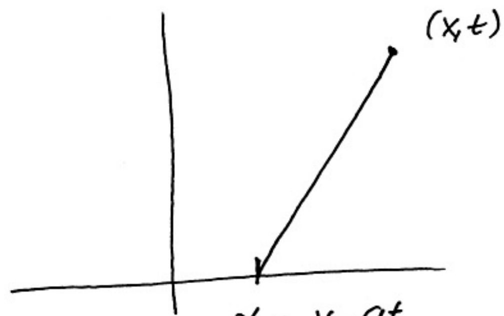
Show MATLAB CODE

and Graphs of the numerical solution.

Domain of Dependence.

(I) $u_t + a u_x = 0$, $\quad u(x,0) = \phi(x)$ $\qquad$ (1)

$a > 0$

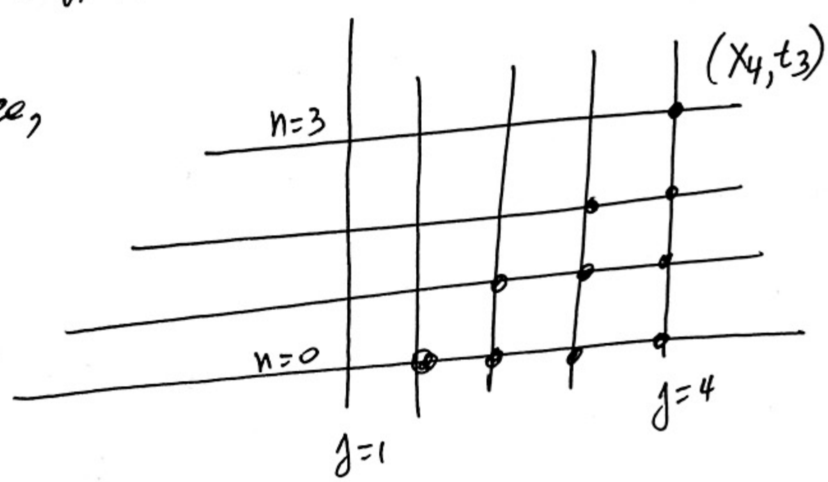


Domain of dep. of IVP (1) at (x,t) is the point x_0. Since this point determine the soln. at (x,t).

(II) FT-BS : $U_j^{n+1} = (1-\alpha) U_j^n + \alpha U_{j-1}^n$ $\qquad$ (2)

Domain of dep. of num. scheme (2) at (x_j, t_n) is $[(j-n)\Delta x, j\Delta x]$ $\qquad$ $\alpha \equiv \frac{c\Delta t}{\Delta x}$

For instance,

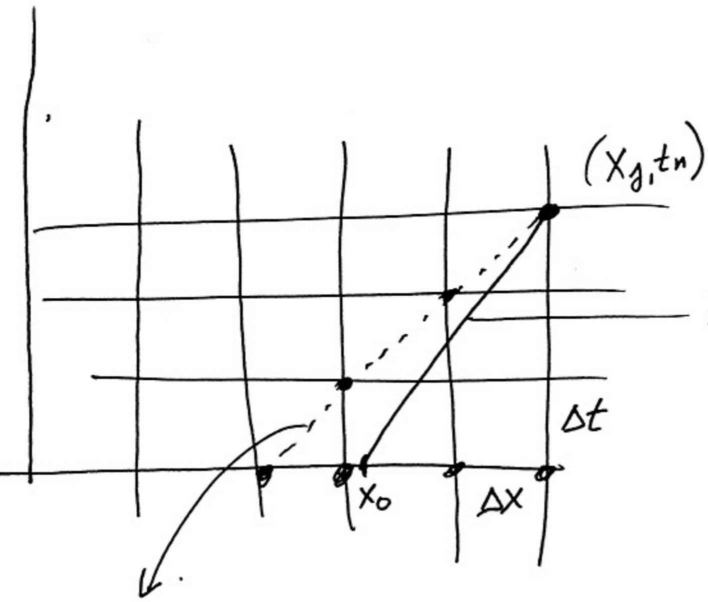


Domain of dep. at (x_4, t_3) of num. scheme (2) is $[\Delta x, 4\Delta x]$

CFL Thm.

A necess. cond. for the converg. of the soln. of a finite diff. approx. to the soln. of (1) for ARBITRARY INITIAL DATA is that

Domain of dep N. Schm $\supset$ Domain of dep. of IVP. (1)



$$x - at = x_j - at_n = x_0$$

$\downarrow$

$$\text{slope} = \frac{1}{a}$$

$$\text{slope} = \frac{\Delta t}{\Delta x}$$

Necessary Condition for Convergence:

Courant number $\quad \frac{1}{a} \geq \frac{\Delta t}{\Delta x} \Rightarrow a \leq \frac{\Delta x}{\Delta t}$

a ↓ phys. wave speed

$\frac{\Delta x}{\Delta t}$ ↓ Num wave speed.

or $\quad \frac{a\Delta t}{\Delta x} \leq 1$

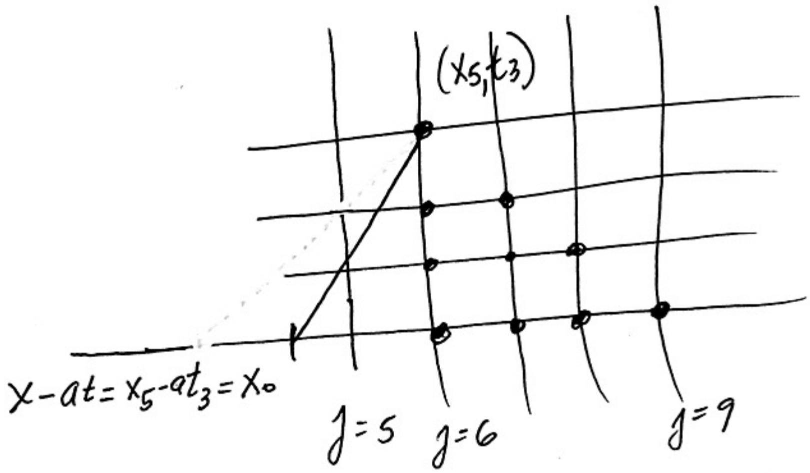
"Domain of dep. of finite difference approx. Contains domain of dependence of IVP (1)."

Logical statement and implications:

Convergence $\Rightarrow$ CFL Cond.

$\neg$ CFL cond $\Rightarrow$ $\neg$ convergence.

FT-FS : $$U_j^{n+1} = (1+\alpha_j^n)\, U_j^n - \alpha_j^n U_{j+1}^n . \qquad (3)$$



Domain of dep. Num. scheme

$[5\Delta x, 9\Delta x]$

$x_0 \notin [5\Delta x, 9\Delta x]$ CFL is not satisfied.

Therefore, Soln. of (3) does not converge to the Soln. of (1). for arbitrary I.C.

CFL Thm can be proved by assuming no CFL and defining an I.C. such that

$$\phi(x) = \begin{cases} 0, & \text{in dom. of dep. N.S. for } (x,t) \\ 1, & \text{in dom of dep. IVP. for } (x,t) \end{cases}$$

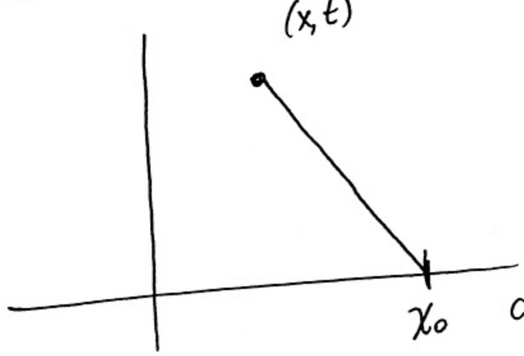
Then, Soln. of N.S. is zero at (x,t), while soln of IVP is nonzero at (x,t).

If we consider the IVP:

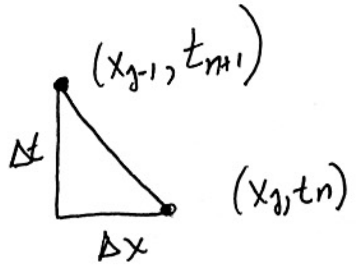
$$u_t + a u_x = 0 , \qquad u(x,0) = \phi(x)$$

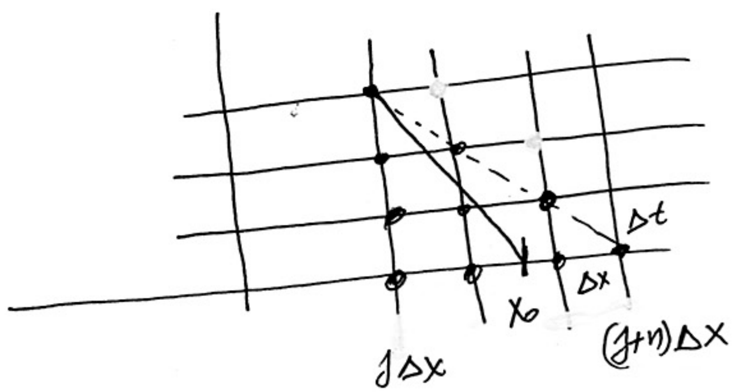
$$a < 0$$

Then



Also, FT-FS scheme (3)





$$\text{Slope} = \frac{t_{n+1} - t_n}{x_{j-1} - x_j} = \frac{\Delta t}{-\Delta x}$$

$$\frac{1}{a} \leq -\frac{\Delta t}{\Delta x}$$

So CFL condition is $x_0 \in [j\Delta x, (j+n)\Delta x]$

which is equivalent to

$$\frac{1}{a} \leq \frac{-\Delta t}{\Delta x} \iff a \geq -\frac{\Delta x}{\Delta t}$$

$$\iff \boxed{\frac{a\Delta t}{\Delta x} \geq -1}$$

Courant #