Iterative Methods Applied to Poisson's Equ.

Continuous:
$$\begin{cases} \nabla^2 u = f(x,y) & (1) \\ u(\vec{x}_s) = \gamma(\vec{x}_s), \quad \vec{x}_s \in \square \end{cases}$$

Discrete 5-point FDM :
$$\boxed{\nabla_5^2 v_{ij} = f_{ij},} \qquad i,j = 1,\dots,m \qquad (2)$$

$$\boxed{\frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{(\Delta x)^2} + \frac{v_{i,j+1} - 2v_{ij} + v_{i,j-1}}{(\Delta y)^2} = f_{ij}} \qquad (3)$$

or

$$\frac{-2(\Delta y^2 + \Delta x^2)}{(\Delta x)^2(\Delta y)^2} v_{ij} + \frac{1}{(\Delta x)^2}(v_{i+1,j} + v_{i-1,j}) + \frac{1}{(\Delta y)^2}(v_{i,j+1} + v_{i,j-1}) = f_{ij}$$

then,
$$v_{i,j} = \frac{(\Delta y)^2}{2(\Delta x^2 + \Delta y^2)}(v_{i+1,j} + v_{i-1,j}) + \frac{(\Delta x)^2}{2(\Delta x^2 + \Delta y^2)}(v_{i,j+1} + v_{i,j-1}) - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} f_{ij}$$

or
$$\boxed{v_{ij} = \theta_x(v_{i+1,j} + v_{i-1,j}) + \theta_y(v_{i,j+1} + v_{i,j-1}) - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} f_{ij}} \qquad (4)$$
$$i,j = 1,\dots,m$$

If $\Delta x = \Delta y \Rightarrow \theta_x = \frac{1}{4}, \quad \theta_y = \frac{1}{4}$

$$\frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} = \frac{(\Delta x)^2}{4}$$

or
$$\boxed{v_{ij} = \frac{1}{4}(v_{i+1,j} + v_{i-1,j}) + \frac{1}{4}(v_{i,j+1} + v_{i,j-1}) - \frac{\overbrace{(\Delta x)^2}^{=h^2}}{4} f_{ij}} \qquad (5)$$

Two methods:

i) <u>Direct</u>: Solve system of equations obtained from (4) or (5)

$$A\vec{U} = \vec{F}$$

where A is given as discussed before.

It can be shown A is nonsingular

ii) <u>Iterative</u>:

a) <u>Jacobi</u>:

$$U_{ij}^{(k+1)} = \frac{1}{4}\left[U_{i+1,j}^{(k)} + U_{i-1,j}^{(k)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k)}\right] - \frac{h^2}{4} f_{ij}$$

It can be shown that it converges for any initial guess: $\vec{U}^{(0)}$.

b) <u>Gauss-Seidel</u>:

$$U_{ij}^{(k+1)} = \frac{1}{4}\left[U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)}\right] - \frac{h^2}{4} f_{ij}$$

Go to general discussion with elementary matrices (3x3).

Complexity:

- A is never storage. Storage is optimal
- Only m^2 in Gauss-Siedel, for solution vector
  $2m^2$ in Jacobi
- Eeach iteration require $\mathcal{O}(m^2)$ work
- \# iterations $= \mathcal{O}(m^2 \log m)$
- Then, work/iter $= \mathcal{O}(m^4 \log m)$

  worst than Gauss elim. $\mathcal{O}(m^3)$ $\longrightarrow \mathcal{O}((m^2)^3) = \mathcal{O}(m^6)$ Not banded.
  with banded solver
- Other with same $\mathcal{O}(m^2)$ work/iter
  converge faster.
- If converging is indep. of h
  then total work $= \mathcal{O}(m^2)$ multigrid methods.
  for many elliptic problems.