Lax Equivalence Theorem.

Definition.– The IVP for the first-order (in time) PDE $u_t = L\,u$. (L differential operator) is well-posed if for any time $T \geq 0$, there is a constant C_T such that any solution $u(x,t)$ satisfies

$$\int_{-\infty}^{\infty} |u(x,t)|^2\,dx \leq C_T \int_{-\infty}^{\infty} |u(0,x)|^2\,dx.$$

for $0 \leq t \leq T$.

Theorem.– A consistent finite difference scheme for a PDE for which the IVP is well-posed is convergent if and only if it is stable.

Proof.– ($\Leftarrow$) Stability $\Rightarrow$ convergence.

Consider the numerical scheme

$$\boxed{\vec{U}^{n+1} = L_\Delta \vec{U}^n} \quad (1)$$

for example, FT-CS heat conduction

$$\underbrace{\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{J-1}^{n+1} \end{bmatrix}}_{=\vec{U}^{n+1}} = \underbrace{\begin{bmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{bmatrix}}_{=L_\Delta} {}^{*} \underbrace{\begin{bmatrix} U_1^n \\ \vdots \\ \vdots \\ U_{J-1}^n \end{bmatrix}}_{=\vec{U}^n}$$

Definition:- (Convergence).

A finite difference approx. $\vec{U}^n$ converges to the solution $\vec{u}^n$ of a PDE on $0 \le t \le T$ in a particular vector norm if

$$\|\vec{u}^n - \vec{U}^n\| \to 0, \text{ when } n \to \infty,\ \Delta x \to 0,\ \Delta t \to 0$$
$$\text{and } n\Delta t \le T$$

Equivalent to say that
for any $\varepsilon > 0$ there is $\delta > 0$ such that
if $\Delta x, \Delta t < \delta$ and $n\Delta t \le T$, T fixed time (obviously $n \to \infty$ as $\Delta t \to 0$)

$$\Rightarrow \quad \|\hat{u}^n - \hat{U}^n\| \le \varepsilon., \quad \text{for all } n \text{ such that } n\Delta t \le T$$

Definition (Consistency)
A finite differ scheme $\vec{U}^{n+1} = L_\Delta \vec{U}^n$ is consistent with a PDE. $\vec{u}^{n+1} = \mathcal{L}\vec{u}^n$ if the local discretization error τ_j^n tends to zero as $\Delta x \to 0$, $\Delta t \to 0$.

Equivalent to say:
For every $\varepsilon > 0$ there is a $\delta > 0$ such that
if $\Delta x, \Delta t < \delta \Rightarrow |\tau_j^n| < \varepsilon$, for all j and n.

Definition (Stability)

A finite difference scheme: $\vec{U}^{n+1} = L_\Delta \vec{U}^n$ of a homogeneous IVP is stable if there exists a positive constant C independent of the mesh spacing $(\Delta t, \Delta x)$ and initial data $\vec{U}_0$, such that

$$\|\vec{U}^n\| \le C\|\vec{U}^0\|, \quad \begin{matrix}\Delta x \to 0 \\ \Delta t \to 0 \Rightarrow n \to \infty\end{matrix} \text{ and } n\Delta t \le \underset{\text{Fixed time.}}{\underset{\downarrow}{T}}$$

Equivalent to say:

There is $\delta > 0$, such that

if $\Delta x, \Delta t < \delta \Rightarrow \|\vec{U}^n\| \le C\|\vec{U}^0\|$, for all n such that $n\Delta t \le T$.

Theorem.-

The F. differ. $\vec{U}^{n+1} = L_\Delta \vec{U}^n$ is stable if and only if there exists a constant C such that

$$\|(L_\Delta)^n\| \le C \quad \text{when} \quad \begin{matrix}\Delta x \to 0 \\ \Delta t \to 0 \Rightarrow n \to \infty\end{matrix} \text{ and } n\Delta t \le \underset{\text{fixed time.}}{\underset{\downarrow}{T}}$$

Proof.- $(\Leftarrow)$ If $\|(L_\Delta)^n\| \le C \Rightarrow \|\vec{U}^n\| = \|L_\Delta \vec{U}^{n-1}\| = \dots = \|(L_\Delta)^n \vec{U}^0\|$

$$\le \|(L_\Delta)^n\| \|\vec{U}^0\| \le C \|\vec{U}^0\|$$

or $\|\vec{U}^n\| \le C\|\vec{U}^0\|$, when $\begin{matrix}\Delta x \to 0 \\ \Delta t \to 0 \Rightarrow n \to \infty\end{matrix}$ and $n\Delta t \le T$.

$(\rightarrow)$ Conversely if $\vec{U}^{n+1} = L_\Delta \vec{U}^n$ is stable

$$\Rightarrow \|\vec{U}^n\| \le C\|\vec{U}^0\| \Leftrightarrow \|(L_\Delta)^n \vec{U}^0\| \le C\|\vec{U}^0\|, \text{ for any } \vec{U}^0.$$

$$\Rightarrow \frac{\|(L_\Delta)^n \vec{U}^0\|}{\|\vec{U}^0\|} \le C, \quad \|\vec{U}^0\| \ne 0 \text{ arbitrary}$$

$$\Rightarrow \|(L_\Delta)^n\| = \max_{\vec{U}^0} \frac{\|(L_\Delta)^n \vec{U}_0\|}{\|\vec{U}^0\|} \le C, \quad \text{for any } \vec{U}^0$$

when $\begin{matrix}\Delta x \to 0 \\ \Delta t \to 0\end{matrix}$ and $n\Delta t \le T$.

Proof.- (Lax Equivalence theorem).

If u_j^n is a solution of the partial diff. equ.

then

$$\vec{u}^{n+1} = L_\Delta \vec{u}^n + \Delta t\, \vec{\tau}^n$$

For example, Heat cond.

$$u_j^{n+1} = r\, u_{j-1}^n - 2r\, u_j^n + r\, u_{j+1}^n + \Delta t\, \tau_j^n$$

where

$$\tau_j^n = -\frac{\Delta t}{2} (u_{tt})_j^{n+\theta} + \sigma \frac{\Delta x^2}{12} (u_{xxxx})_{j+\xi}^n$$

or

$$\boxed{\vec{u}^{n+1} = L_\Delta \vec{u}^n + \Delta t\, \vec{\tau}^n.} \qquad (2)$$

The difference of the vector solution $\vec{u}^n$ of the PDE.

and the vector solution $\vec{v}^n$ of the discrete approx.

is called $\vec{e}^n = \vec{u}^n - \vec{v}^n$ (global discretization error)

Subtracting (1.) from (2)

$$\vec{e}^{n+1} = \vec{u}^{n+1} - \vec{v}^{n+1} = L_\Delta (\vec{u}^n - \vec{v}^n) + \Delta t\, \vec{\tau}^n = L_\Delta \vec{e}^n + \Delta t\, \vec{\tau}^n.$$

$$\Rightarrow \quad \boxed{\vec{e}^{n+1} = L_\Delta \vec{e}^n + \Delta t\, \vec{\tau}^n} \qquad (3)$$

If L_Δ is independent of n by iterating on (3)

$$\vec{e}^{\,n} = L_\Delta \vec{e}^{\,n-1} + \Delta t\, \vec{\tau}^{\,n-1} = L_\Delta \left(L_\Delta \vec{e}^{\,n-2} + \Delta t\, \vec{\tau}^{\,n-2} \right) + \\ + \Delta t\, \vec{\tau}^{\,n-1}$$

$$= L_\Delta^2 \vec{e}^{\,n-2} + \Delta t \left[L_\Delta \vec{\tau}^{\,n-2} + \vec{\tau}^{\,n-1} \right] =$$

$$= L_\Delta^3 \vec{e}^{\,n-3} + \Delta t \left[L_\Delta^2 \vec{\tau}^{\,n-3} + L_\Delta \vec{\tau}^{\,n-2} + \vec{\tau}^{\,n-1} \right] =$$

$$\cdots\cdots = L_\Delta^n \vec{e}^{\,0} + \Delta t \left[L_\Delta^{n-1} \vec{\tau}^{\,0} + L_\Delta^{n-2} \vec{\tau}^{\,1} + L_\Delta^{n-3} \vec{\tau}^{\,2} + \cdots \\ + \cdots + L_\Delta^{n-n} \vec{\tau}^{\,n-1} \right]$$

or

$$\boxed{\vec{e}^{\,n} = L_\Delta^n \vec{e}^{\,0} + \Delta t \left[L_\Delta^{n-1} \vec{\tau}^{\,0} + L_\Delta^{n-2} \vec{\tau}^{\,1} + \cdots + \vec{\tau}^{\,n-1} \right]}$$

$$\Rightarrow \|\vec{e}^{\,n}\| \le \Delta t \left[\|L_\Delta^{n-1}\| \, \|\vec{\tau}^{\,0}\| + \|L_\Delta^{n-2}\| \, \|\vec{\tau}^{\,1}\| + \cdots + \|\vec{\tau}^{\,n-1}\| \right] \quad (4)$$

Let's choose a time of interest T and an arbitrary $\varepsilon > 0$. Since numerical scheme is consistent there exists $\delta_1 > 0$ such that $\|\vec{\tau}^{\,k}\| \le \varepsilon$ if $\Delta t, \Delta x < \delta_1$ for all k such that $k\Delta t \le T$!

Also, using the hypothesis that the scheme is stable and the previous theorem about stability we conclude that there exist C and $\delta_2 > 0$ such that

if $\Delta x, \Delta t < \delta_2 \Rightarrow \|(L_\Delta)^k\| \leq C$ for all k such that $k\Delta t \leq T$.

Therefore, choosing $\delta = \min(\delta_1, \delta_2)$, for $\Delta x, \Delta t < \delta$

(4) reduces to

$$\|\hat{e}^n\| \leq \Delta t\left((n-1)C\varepsilon + \varepsilon\right)\begin{cases} \leq \Delta t\, n C \varepsilon, & \text{if } C \geq 1. \\ < \Delta t\, n \varepsilon, & \text{if } C < 1 \end{cases}$$

Since $n\Delta t \leq T$

$$\Rightarrow \quad \|\hat{e}^n\| \leq \begin{cases} TC\varepsilon, & C \geq 1 \\ \quad\text{or} & \\ T\varepsilon, & C < 1 \end{cases}$$

In both cases, the scheme converges.