3.3 Matrix Stability Analysis

Used to analyze more general linear problems

Recall that $\boxed{U_j^{n+1} = r U_{j-1}^n + (1-2r) U_j^n + r U_{j+1}^n}$.

FT-CS for heat conduction IBVP with

B.C.: $u(0,t) = u(1,t) = 0$, for all t.

The matrix form of this numerical scheme is given by

$$\begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_J^{n+1} \end{pmatrix} = \begin{pmatrix} 1-2r & r & 0 & 0 & \cdots & 0 \\ r & 1-2r & r & 0 & \cdots & 0 \\ 0 & r & 1-2r & r & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & r & 1-2r \end{pmatrix} \begin{pmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{J-1}^n \end{pmatrix} \quad (1.0)$$

or briefly

$$\boxed{\vec{U}^{n+1} = L_\Delta \vec{U}^n} \quad (1.1)$$

$r = \sigma \Delta t / \Delta x^2$ → constant.

We will consider numerical schemes represented by (1.1)

If as in FT-CS for Heat cond. L_Δ is indep. of time level n, after iteration we obtain

$$\vec{U}^n = (L_\Delta)^n \vec{U}^0$$

Taking norm

$$\|\vec{U}^n\| = \|(L_\Delta)^n \vec{U}^0\| \le \|L_\Delta^n\| \, \|\vec{U}^0\|$$

According to our definition of stability (1.1) is stable if and only if there exists C (constant) such that

$$\|(L_\Delta)^n\| \le C, \quad n \to \infty \quad \begin{matrix}\Delta x \to 0 \\ \Delta t \to 0\end{matrix}, \quad n\Delta t \le T.$$

Recall, that for a matrix

$$A = (a_{ij})$$

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

For FT-CS Heat cond.

$$\|L_\Delta\|_\infty = |r| + |1-2r| + |r| = r + |1-2r| + r$$

if $r \le 1/2 \Rightarrow \|L_\Delta\|_\infty = 1$

and as a consequence

$$\|\vec{U}^n\| \le \|\vec{U}^0\|,$$

which proves that this scheme is stable in the max norm.

In general, for linear difference schemes as (1.1), we have the following theorem.

<u>Theo 3.3.2.</u> A sufficient condition for stability of the linear finite difference scheme (1.1) is that there exists a constant c, independent of Δx and Δt, such that

$$\|L_\Delta\| \le 1 + c\Delta t. \qquad * (3.1)$$

<u>Proof.-</u> Assuming L_Δ is independent of n

$$\|\vec{U}^n\| \le \|(L_\Delta)^n\| \, \|\vec{U}^0\| \le \|L_\Delta\|^n \|\vec{U}^0\|$$

Then using (3.1)

$$\|\vec{U}^n\| \le (1 + c\Delta t)^n \|\vec{U}^0\| \le e^{cn\Delta t} \|\vec{U}^0\|$$

(using that $1 + z \le e^z$)

If we define T as the max. time in the computation, then $n\Delta t \le T$ and

$$\|\vec{U}^n\| \le e^{cT} \|\vec{U}^0\|$$

$$\underset{C \text{ (stability constant)}}{\overset{|||}{}}$$

Applications:

Example 3.3.2.- Consider the finite difference scheme

$$v_j^{n+1} = C_{-1} v_{j-1}^n + C_1 v_{j+1}^n \qquad j = 0, 1, .., J-1 \qquad (4.1)$$

with periodic initial Condition. Period $= J$

Lax - Friedrichs is a particular case of (4.1) with

$$C_{-1} = \frac{1+\alpha}{2}, \quad C_1 = \frac{1-\alpha}{2}$$

Clearly, (4.1) can be written as

$$\vec{v}^{n+1} = L_\Delta \vec{v}^n$$

periodicity

$$v_0^{n+1} = C_{-1} v_{J-1} + C_1 v_1^n$$

$$v_1^{n+1} = C_{-1} v_0 + C_1 v_2^n$$

$$\vdots$$

$$v_{J-1}^{n+1} = C_{-1} v_{J-2}^n + C_1 v_J^n = C_{-1} v_{J-2}^n + C_1 v_0^n$$

Therefore,

$$L_\Delta = \begin{pmatrix} 0 & c_1 & 0 & 0 & 0 & \cdots & c_{-1} \\ c_{-1} & 0 & c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_{-1} & 0 & c_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & 0 \\ 0 & \cdots & 0 & c_{-1} & 0 & & c_1 \\ c_1 & 0 & \cdots & 0 & c_{-1} & & 0 \end{pmatrix}, \qquad \vec{v}^{\,n} = \begin{pmatrix} v_0^n \\ v_1^n \\ \vdots \\ v_{J-1}^n \end{pmatrix}$$

If $|c_{-1}| + |c_1| \le 1$, then $\|L_\Delta\|_\infty \le 1$

Thus, (4.1) is stable in the max norm. when

$$\left|\frac{1+\alpha}{2}\right| + \left|\frac{1-\alpha}{2}\right| \le 1$$

or $\frac{1}{2}\left(|1+\alpha| + |1-\alpha|\right) \le 1$, which is true if $|\alpha| \le 1$

This stability result was obtained before by applying the max pple.

Lax-Wendroff scheme for

$$u_t + a u_x = 0$$

$$u_j^{n+1} = u_j^n + \Delta t\, (u_t)_j^n + \frac{\Delta t^2}{2} (u_{tt})_j^n + \frac{\Delta t^3}{3!} (u_{ttt})_j^n + \dots$$

Using the equ. $u_t = -a u_x$

$$u_{tt} = -a u_{xt} = -a (u_t)_x = -a(-a u_x)_x = \\ = a^2 u_{xx}$$

Subst. in the Taylor expansion

$$u_j^{n+1} = u_j^n + \Delta t\, (-a u_x)_j^n + \frac{\Delta t^2}{2} (a^2 u_{xx})_j^n + \mathcal{O}(\Delta t^3)$$

$$= u_j^n - a \Delta t\, (u_x)_j^n + \frac{a^2 \Delta t^2}{2} (u_{xx})_j^n + \mathcal{O}(\Delta t^3)$$

$$= u_j^n - a \frac{\Delta t}{2\Delta x} \left(u_{j+1}^n - u_{j-1}^n \right) + \frac{a^2 \Delta t^2}{2\Delta x^2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) + \mathcal{O}(\Delta t^3)$$

$$\Rightarrow U_j^{n+1} = U_j^n - \frac{\alpha}{2} \left(U_{j+1}^n - U_{j-1}^n \right) + \frac{\alpha^2}{2} \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n \right)$$

or

$$\boxed{U_j^{n+1} = \frac{\alpha + \alpha^2}{2} U_{j-1}^n + (1 - \alpha^2) U_j^n + \frac{\alpha^2 - \alpha}{2} U_{j+1}^n} \quad (6.1)$$

local error $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$

Check this!

Max pple. criteria does not apply for L-W scheme

Since,

if $0<\alpha<1$, $\frac{\alpha^2-\alpha}{2}<0$

if $|\alpha|>1$, $1-\alpha^2<0$

if $-1<\alpha<0$, $\frac{\alpha+\alpha^2}{2}<0$

To prove Stability of (6.1), we will use Euclidean or l_2-norm. and will try to satisfy the stability condition directly.

The procedure is as follows:

① Square (6.1)

$$\left(v_j^{n+1}\right)^2 = \left[\frac{\alpha(\alpha+1)}{2} v_{j-1}^n + (1-\alpha^2) v_j^n + \frac{\alpha(\alpha-1)}{2} v_{j+1}^n\right]^2 \quad (7.1)$$

② Assume $|\alpha| \le 1$ and add

$$\frac{\alpha^2(1-\alpha^2)}{4}\left[v_{j-1}^n - 2v_j^n + v_{j+1}^n\right]^2 \geq 0$$

③ then

$$\left(v_j^{n+1}\right)^2 \le \text{rhs } (7.1) + \frac{\alpha^2(1-\alpha^2)}{4}\left[\quad\right]^2$$

4 Some terms will cancel (show my notes).

5 Sum over a period and get

$$\sum_{j=0}^{J-1} \left(v_j^{n+1}\right)^2 \le \sum_{j=0}^{J-1} \left[\frac{\alpha^2(1+\alpha)}{2} \left(v_{j-1}^n\right)^2 + (1-\alpha^2)\left(v_j^n\right)^2 + \frac{\alpha^2(1-\alpha)}{2}\left(v_{j+1}^n\right)^2 - \alpha(1-\alpha^2)\left(v_{j+1}^n v_j^n - v_j^n v_{j-1}^n\right) \right] \quad (8.1)$$

6 Reindex the summation and use periodicity: $v_{-1}^n = v_{J-1}^n$
$v_0^n = v_J^n$.

then (8.1) simplifies to

$$\sum_{j=0}^{J-1} \left(v_j^{n+1}\right)^2 \le \sum_{j=0}^{J-1} \left[\frac{\alpha^2(1+\alpha)}{2} + (1-\alpha^2) + \frac{\alpha^2(1-\alpha)}{2} \right] \left(v_j^n\right)^2 =$$

$$= \sum_{j=0}^{J-1} \left(v_j^n\right)^2 .$$

$$\Rightarrow \quad \|\vec{v}^{\,n+1}\|_2 \le \|\vec{v}^{\,n}\|_2 \quad \text{if} \quad |\alpha| \le 1$$

Courant, Friedrichs, Levy Condition.

$$\frac{\alpha^2(1+\alpha)^2}{4}\left(v_{j-1}^n\right)^2 + \frac{\alpha^2(1-\alpha^2)}{4}\left(v_{j-1}^n\right)^2 = \frac{\alpha^2(1+\alpha)}{2}\left(v_{j-1}^n\right)^2$$

$$+ (1-\alpha^2)^2\left(v_j^n\right)^2 + \alpha^2(1-\alpha^2)\left(v_j^n\right)^2$$

$$+ \frac{\alpha^2(\alpha-1)^2}{4}\left(v_{j+1}^n\right)^2 + \frac{\alpha^2(1-\alpha^2)}{4}\left(v_{j+1}^n\right)^2$$

$$+ \alpha(\alpha+1)(1-\alpha^2)\, v_{j-1}^n v_j^n$$

$$+ \frac{\alpha^2(\alpha+1)(\alpha-1)}{2}\, v_{j-1}^n v_{j+1}^n + \frac{\alpha^2(1-\alpha^2)}{2}\, v_{j-1}^n v_{j+1}^n = 0$$

$$+ \alpha(\alpha-1)(1-\alpha^2)\, v_{j+1}^n v_j^n$$

$$\frac{1}{4}\alpha^2(1+\alpha)\left[(1+\cancel{\alpha}) + (1-\cancel{\alpha})\right] = \frac{\alpha^2(1+\alpha)}{2}$$

$$\left[v_{j-1}^n - 2v_j^n + v_{j+1}^n\right]^2 = \left(v_{j-1}^n\right)^2 + 4\left(v_j^n\right)^2 + \left(v_{j+1}^n\right)^2 +$$

$$+ 2v_{j-1}^n v_{j+1}^n - 4v_{j-1}^n v_j^n - 4v_j^n v_{j+}^n$$