4.3 Multilevel Schemes and the Method of Lines.

Again consider heat conduction equation

$$u_t = \sigma u_{xx}$$

and construct CT-CS numerical scheme.

$$(u_t)_j^n - \sigma (u_{xx})_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \sigma \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

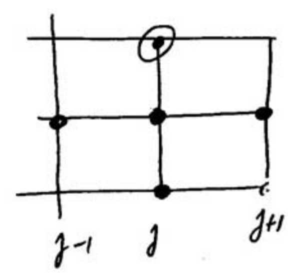
$$+ \underbrace{\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)}_{\tau_j^n}$$

therefore, scheme is given by

$$\boxed{v_j^{n+1} = v_j^{n-1} + 2r\left(v_{j-1}^n - 2v_j^n + v_{j+1}^n\right)} \quad (1.1)$$

It's called Leap Frog method.

Explicit



Stability (Von Neumann Analysis).

Assuming periodic initial data on $[0, 2\pi]$.

$$U_j^n = \sum_{k=1}^{J-1} A_k^n W_j^k , \qquad W_j^k = e^{i 2\pi k j/J}$$

Subst. in (1.1). + orthog.

$$A_k^{n+1} = A_k^{n-1} + 2r \left(e^{-2\pi i k/J} - 2 + e^{2\pi i k/J} \right) A_k^n$$

$$e^{-2\pi i k/J} + e^{2\pi i k/J} = 2 \cos(2\pi k/J)$$

$$\Rightarrow \quad A_k^{n+1} = A_k^{n-1} + 4r \left(\cos(2\pi k/J) - 1 \right) A_k^n$$

$$\therefore \quad \boxed{A_k^{n+1} = A_k^{n-1} - 8r \sin^2(k\pi/J) A_k^n} \qquad (2.1)$$

Discrete Equation (2.1) can be solved assuming a solution

as $\quad A_k^n = (M_k)^n C_k , \qquad C_k$ is a constant.

Subst. in (2.1) leads to

$$\boxed{(M_k)^{n+1} = (M_k)^{n-1} - 4r \mu_k (M_k)^n} \qquad (2.2)$$

Where $\quad \mu_k \equiv 2 \sin^2(k\pi/J)$.

Dividing by $(M_k)^{n-1}$ (2.2).

$$(M_k)^2 + 4r\mu_k (M_k) - 1 = 0$$

$$\Rightarrow \quad M_k^{\pm} = -2r\mu_k \pm \sqrt{1+(2r\mu_k)^2} \qquad (3.1)$$

Therefore two linearly indep. soln. of (2.2) are

$$(A_k^n)^+ = C_k^+ (M_k^+)^n, \quad (A_k^n)^- = C_k^- (M_k^-)^n$$

and the general soln. can be written as

$$A_k^n = C_k^+ (M_k^+)^n + C_k^- (M_k^-)^n$$

C_k^+ and C_k^- are obtained from I.C.

To understand better $M_k^{\pm}$, square root in (3.1) is expanded in Taylor series. assuming $\alpha r\mu_k \ll 1$

$$\left(1+(2r\mu_k)^2\right)^{1/2} = 1 + \frac{1}{2}(2r\mu_k)^2 + \cdots$$

Therefore,

$$M_k^+ = -2r\mu_k + 1 + \mathcal{O}(r^2\mu_k^2) = 1 - 2r\mu_k + \mathcal{O}(r^2\mu_k^2)$$

$$M_k^- = -\left[1 + 2r\mu_k + \mathcal{O}(r^2\mu_k^2)\right]$$

Obviously, $M_k^+ \downarrow$ when $n \uparrow$ since $1-2r\mu_k < 1$ for $r\mu_k \ll 1$.

and M_k^- oscilates and increases in amplitude $1+2r\mu_k > 1$.

Therefore, Leap Frog is <u>unstable</u>.

<u>Du Fort and Frankel Scheme:</u>

Improve Leap Frog stability by using an average of v_j^{n-1} and v_j^{n+1} instead of v_j^n

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = \sigma \frac{v_{j-1}^n - 2\left(\frac{v_j^{n+1}+v_j^{n-1}}{2}\right) + v_{j+1}^n}{\Delta x^2}$$

or $$\boxed{(1+2r)v_j^{n+1} = (1-2r)v_j^{n-1} + 2r\left(v_{j-1}^n + v_{j+1}^n\right)}$$

In a Hwk problem for this section you are asked to prove that Du Fort-Frankel scheme is unconditionally stable in L_2 norm.

Is it possible for an explicit scheme to be unconditionally stable?

No, if it is consistent.

Problem in this case is that Du Fort-Frankel to be consistent with heat cond. equ. requires that $\Delta t \to 0$ at a faster rate than $\Delta x \to 0$

in other words $\Delta t \ll \Delta x$

If $\Delta t \simeq \Delta x$ or $\frac{\Delta t}{\Delta x} = \mu$ constant.

then Du Fort-Frankel is consistent with

$$u_t - \sigma u_{xx} + \sigma \mu^2 u_{tt} = 0$$

and for DF-F scheme to be consistent with Heat cond. equation $\frac{\Delta t}{\Delta x^2} \simeq$ constant