

## Acoustic Scattering from an Infinite Circular Cylindrical Obstacle.

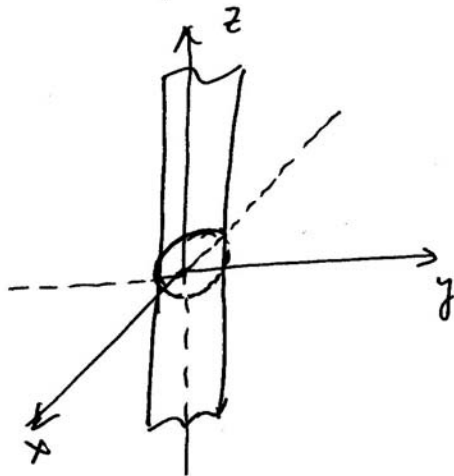
In the previous section, we showed that under the hypothesis of small disturbance from an equilibrium state, the leading order pressure term satisfies the wave equation.

$$p_{tt} = c^2 \nabla^2 p.$$

In this section, we want to study the scattering of an incident plane wave.

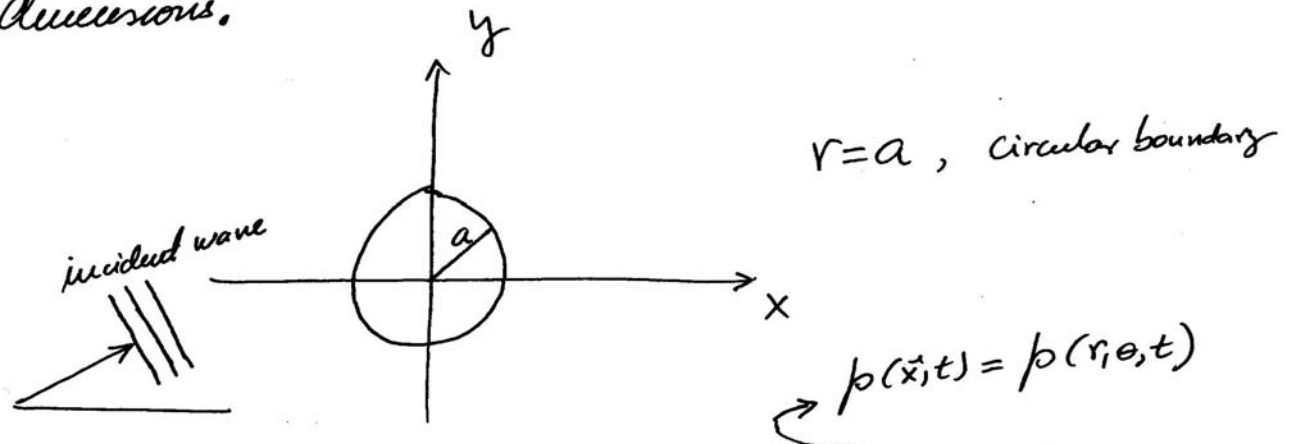
$$p_{\text{inc}}(\vec{x}, t) = e^{ik(x \cos \alpha + y \sin \alpha)} e^{-i\omega t}$$

propagating in a direction parallel to the  $xy$ -plane from an infinite circular cylindrical obstacle.

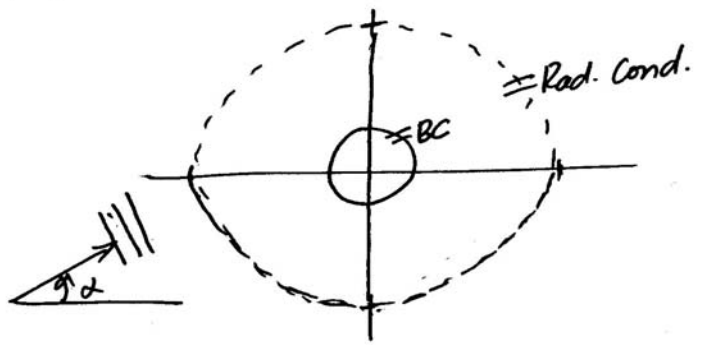


(2)

If the axis of the Cylinder is the  $z$ -axis  
 the scattered wave is the same in every plane  
 $Z \equiv K(\text{const.})$ . Therefore, the problem reduces to only two  
 dimensions.



It's convenient to use polar coordinates due to the  
 circular shape of the obstacle. Also, we will need  
 to impose radiation conditions at  $\infty$  when  $r \rightarrow \infty$ .  
 Therefore, we will need to solve an exterior problem  
 in an infinite domain



(3)

Depending on the nature of the obstacle, we will have two different types of BC's:

$$\boxed{p(a, \theta, t) = 0,}$$
$$\boxed{\frac{\partial p}{\partial r}(a, \theta, t) = 0,}$$

Soft obstacle or <sup>empty</sup> cavity.

Hard obstacle  
(rigid body)

Also, for the problem to be well-posed we need to impose a radiation condition at infinity for the scattered pressure,  $p_{sc}(r, \theta, t)$ .

$p_{sc}$  is defined as the difference

$$\boxed{p_{sc}(r, \theta, t) = p(r, \theta, t) - p_{in}(r, \theta, t).}$$

The radiation condition at  $\infty$  is due to Sommerfeld and is given by

$$\boxed{\frac{\partial p_{sc}}{\partial t} + c \frac{\partial p_{sc}}{\partial r} \xrightarrow{r \rightarrow \infty} 0}$$

(4)

Therefore, the IBVP. is defined as (empty cavity).

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla_{r,\theta}^2 p = c^2 \left[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} \right] \quad (1)$$

$$\text{B.C. } p(a, \theta, t) = 0 \quad (2)$$

$$\text{IC's. } \begin{cases} p(r, \theta, 0) = f(r, \theta) \\ p_t(r, \theta, 0) = g(r, \theta) \end{cases} \quad (3)$$

$$\text{Rad. Cond. } \frac{\partial p_{sc}}{\partial t} + c \frac{\partial p_{ss}}{\partial r} \rightarrow 0 \quad (4)$$

The incident wave in polar coordinates

can be written as

$$p_{inc}(r, \theta, t) = e^{ikr \cos(\theta - \alpha)} e^{-i\omega t}$$

In fact,

$$\begin{aligned} p_{inc}(x, y, t) &= e^{ik(x \cos \alpha + y \sin \alpha)} e^{-i\omega t} \\ &= e^{ik(r \cos \theta \cos \alpha + r \sin \theta \sin \alpha)} e^{-i\omega t} \\ &= e^{ikr \cos(\theta - \alpha)} e^{-i\omega t} \end{aligned}$$

(5)

It can be easily proved that the incident wave satisfies the wave equation (part of the midterm project).

Therefore, it is convenient to formulate the above IBVP in terms of  $p_{sc}$ . Recall that  $p = p_{inc} + p_{sc}$ .

$$\left\{ \begin{array}{l} \frac{\partial^2 p_{sc}}{\partial t^2} = c^2 \left[ \frac{\partial^2 p_{sc}}{\partial r^2} + \frac{1}{r} \frac{\partial p_{sc}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p_{sc}}{\partial \theta^2} \right] \quad (5.1) \end{array} \right.$$

$$p_{sc}(a, \theta, t) = -p_{in}(a, \theta, t) = -e^{+ika \cos(\theta - \alpha)} e^{-i\omega t} \quad (5.2)$$

$$p_{sc}(r, \theta, 0) = \tilde{f}(r, \theta) \quad (5.3)$$

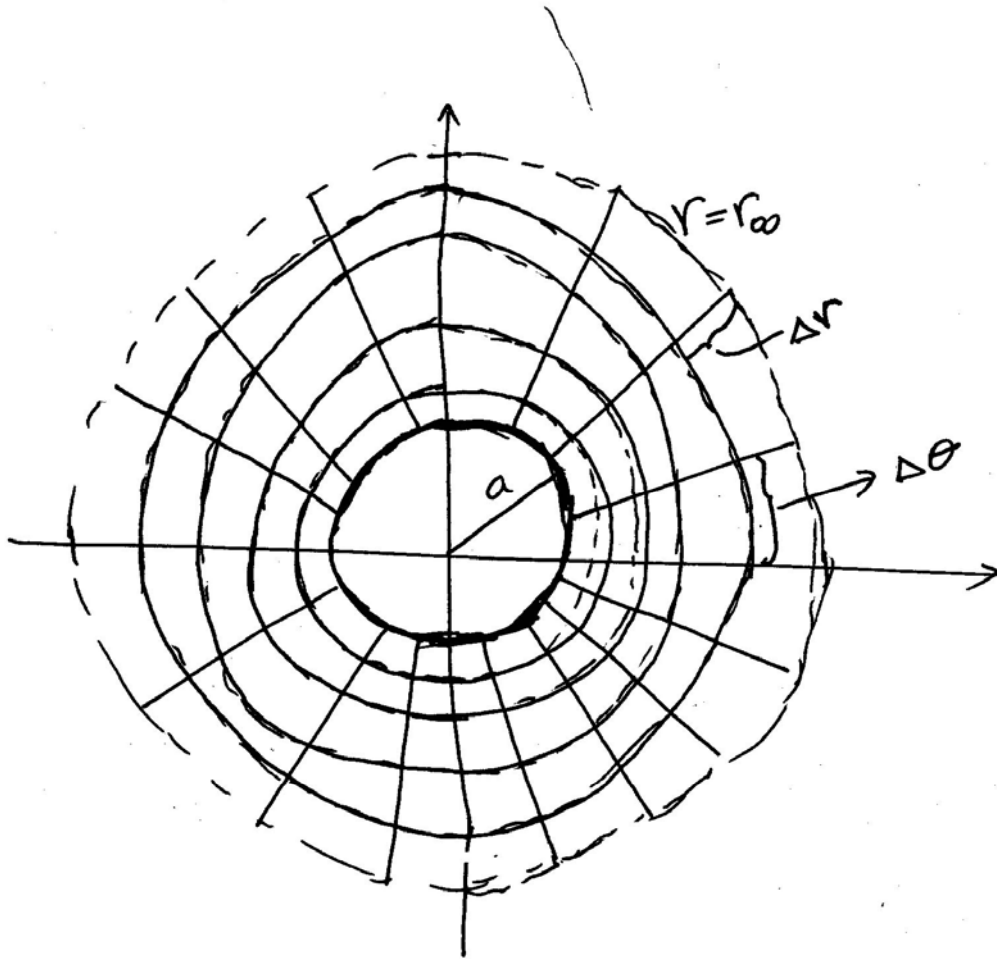
$$(p_{sc})_t(r, \theta, 0) = \tilde{g}(r, \theta) \quad (5.4)$$

$$\frac{\partial p_{sc}}{\partial t} + c \frac{\partial p_{sc}}{\partial r} \rightarrow 0 \quad (5.5)$$

Since the incident wave is harmonic in time, the limiting amplitude principle establishes that the scattered pressure wave  $p_{sc}$  has also time harmonic dependence. It means that after certain time the solution for the above IBVP will reach time harmonic steady state regardless of the IC's.

⑥

We now define a partition of our  
Computational domain



$$r_i = a + i * \Delta r, \quad i = 0, 1, 2, \dots, N.$$

$$\theta_j = j * \Delta \theta, \quad j = 0, \dots, M$$

$$\theta_0 = 0, \quad \theta_M = 2\pi.$$

②

Numerical Solution of IBVP (5) for the Scattered pressure.

(I) Using centered differes to approximate all the derivatives involved, we obtain

$$\frac{(p_{sc})_{ij}^{n+1} - 2(p_{sc})_{ij}^n + (p_{sc})_{ij}^{n-1}}{\Delta t^2} = c^2 \left[ \frac{(p_{sc})_{i+1,j}^n - 2(p_{sc})_{ij}^n + (p_{sc})_{i-1,j}^n}{\Delta r^2} + \frac{1}{r_i} \frac{(p_{sc})_{i+1,j}^n - (p_{sc})_{i-1,j}^n}{2\Delta r} + \frac{1}{r_i^2} \frac{(p_{sc})_{i,j+1}^n - 2(p_{sc})_{ij}^n + (p_{sc})_{i,j-1}^n}{\Delta \theta^2} \right]$$

Solving for  $(p_{sc})_{ij}^{n+1}$ , we obtain the multilevel CT-CS scheme

$$(p_{sc})_{ij}^{n+1} = \left[ 2 - 2\left(\frac{c\Delta t}{\Delta r}\right)^2 - 2\left(\frac{c\Delta t}{r_i\Delta\theta}\right)^2 \right] (p_{sc})_{ij}^n + \left[ \left(\frac{c\Delta t}{\Delta r}\right)^2 + \frac{c^2\Delta t^2}{2r_i\Delta r} \right] (p_{sc})_{i+1,j}^n + \left[ \left(\frac{c\Delta t}{\Delta r}\right)^2 - \frac{c^2\Delta t^2}{2r_i\Delta r} \right] (p_{sc})_{i-1,j}^n + \left(\frac{c\Delta t}{r_i\Delta\theta}\right)^2 \left[ (p_{sc})_{i,j+1}^n + (p_{sc})_{i,j-1}^n \right] + (p_{sc})_{ij}^{n-1}$$

(7)

(II) The radiation condition is applied exactly at a fictitious infinite boundary  $r = r_{\infty}$ .

(You have to choose an appropriate value for  $r_{\infty}$ ).

Then the radiation condition in discrete form reduces to (using centered diffes).

$$\frac{(p_{sc})_{N,j}^{n+1} - (p_{sc})_{N,j}^{n-1}}{2\Delta t} + c \frac{(p_{sc})_{N+1,j}^n - (p_{sc})_{N-1,j}^n}{2\Delta r} = 0.$$

$(p_{sc})_{N+1,j}^n$  represent the value of pressure at a ghost point.  $(r_{N+1}, \theta_j)$ .

(III) The B.C. at the obstacle boundary is given by

$$(p_{sc})_0^n = -e^{+ika \cos(\theta_j - \alpha)} e^{-i\omega t n}, \quad \begin{matrix} j = 0, \dots, M \\ n = 1, 2, \dots, N_T \end{matrix}$$

(IV) I.C's are arbitrary. For example

$$(p_{sc})(r_i, \theta_i, 0) = 0$$

$$(p_{sc})_t(r_i, \theta_i, 0) = 0$$

8

### Exact Solution

The limiting amplitude pp/e establishes that

$$\lim_{t \rightarrow \infty} p_{sc}(r, \theta, t) = \bar{p}_{sc}(r, \theta) e^{-i\omega t}$$

Subst. of  $p_{sc}(r, \theta, t)$  by  $\bar{p}_{sc}(r, \theta) e^{-i\omega t}$  in equs. (5) leads to

$$-\omega^2 \bar{p}_{sc} = c^2 \nabla_{r,\theta}^2 \bar{p}_{sc} e^{-i\omega t}$$

$$\Rightarrow \left[ \nabla_{r,\theta}^2 \bar{p}_{sc} + \frac{\omega^2}{c^2} \bar{p}_{sc} \right] e^{-i\omega t} = 0$$

$$\underline{c = \omega/k}$$

$$\text{or } \boxed{\nabla_{r,\theta}^2 \bar{p}_{sc} + k^2 \bar{p}_{sc} = 0.}$$

Also, the B.C. reduces to

$$\bar{p}_{sc}(a, \theta) e^{-i\omega t} = -\bar{p}_{in}(a, \theta) e^{-i\omega t} = -e^{+ika \cos(\theta - \alpha)} e^{-i\omega t}$$

$$\Rightarrow \boxed{\bar{p}_{sc}(a, \theta) = -e^{+ika \cos(\theta - \alpha)}}$$

(9)

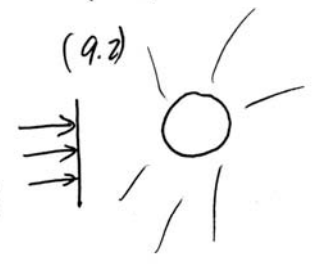
and the radiation condition reduces to

$$\left[ -i\omega \bar{p}_{sc}(r, \theta) + c \frac{\partial \bar{p}_{sc}}{\partial r} \right] e^{-i\omega t} \xrightarrow{r \rightarrow \infty} 0 \quad c = \omega/k$$

or  $\boxed{\frac{\partial \bar{p}_{sc}}{\partial r} - ik \bar{p}_{sc} \xrightarrow{r \rightarrow \infty} 0}$

Therefore, when  $t \rightarrow \infty$  IBVP (5) is equivalent to the BVP.

$$\begin{cases} \nabla_{r,\theta}^2 \bar{p}_{sc} + k^2 \bar{p}_{sc} = 0 & (9.1) \\ \bar{p}_{sc}(a, \theta) = -e^{+ika \cos(\theta-d)} & (9.2) \\ \frac{\partial \bar{p}_{sc}}{\partial r} - ik \bar{p}_{sc} \xrightarrow{r \rightarrow \infty} 0 & (9.3) \end{cases}$$



$$p_{inc}(r, \theta, t) = \bar{p}_{inc}(r, \theta) e^{-i\omega t} = e^{+ikr \cos(\theta-d)} e^{-i\omega t} \quad (9.4)$$

It's exact solution can be computed by the method of Separation of Variables and is given by

$$\boxed{p_{sc}(r, \theta) = \sum_{m=0}^{\infty} A_m H_m^{(1)}(kr) \cos m\theta} \quad (9.5)$$

Where  $H_m^{(1)}(kr)$  are cylindrical Hankel functions of the first kind of order  $m$ .

These Hankel fns. behave at  $\infty$  as

$$\begin{aligned} H_0^{(1)}(kr) &\xrightarrow{kr \rightarrow \infty} \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(kr)^{1/2}} e^{i(kr - \pi/4)} \\ \vdots & \\ H_m^{(1)}(kr) &\xrightarrow{kr \rightarrow \infty} (i)^m \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr - \pi/4)} \end{aligned}$$

To determine the coefficients  $A_m$ 's, we use the B.C. at the obstacle. But first, we need to express

$$\begin{aligned} \bar{\phi}_{\text{inc}}(kr, \theta) &= e^{i kr \cos \theta} \\ &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta, \end{aligned}$$

if  $\underline{\alpha=0}$   
plane wave propag.  
in the positive x-direct.

where  $\epsilon_0 = 1, \epsilon_m = 2, \text{ for } m > 0.$

Then using B.C. at  $\underline{r=a}$ .

$$\sum_{m=0}^{\infty} A_m H_m^{(1)}(ka) \cos m\theta = - \sum_{m=0}^{\infty} \epsilon_m i^m J_m(ka) \cos m\theta$$

$$\therefore A_m = - \frac{\epsilon_m i^m J_m(ka)}{H_m^{(1)}(ka)}$$

(11)

Therefore, the exact soln. of our BVP for the scattered pressure wave is given by

$$\bar{p}_{sc}(r, \theta) = \sum_{m=0}^{\infty} \frac{-\epsilon_m i^m J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr) \cos m\theta.$$

Differential Scattering Cross Section. Far Field Solution.

In 1956, Wilcox showed that the solution

$$\bar{p}_{sc}(r, \theta) \rightarrow w(r, \theta) \frac{e^{ikr}}{r^{1/2}},$$

where

$$w(r, \theta) \equiv \sum_{j=0}^{\infty} \frac{A_j(\theta)}{r^j}$$

It means

$$\bar{p}_{sc}(r, \theta) \rightarrow \frac{e^{ikr}}{r^{1/2}} \left[ A_0(\theta) + \frac{A_1(\theta)}{r} + \dots \right] \quad (11.1)$$

The leading order coefficient in this expansion is called the differential Scatt. Cross section or far field amplitude.

We can also write it including time as

$$p_{sc}(r, \theta, t) \rightarrow \frac{e^{ikr}}{r^{1/2}} \left[ A_0(\theta) + \frac{A_1(\theta)}{r} + \dots \right] e^{-i\omega t} \quad (12.1)$$

Neglecting terms of  $O(1/r^{3/2})$  and solving for  $A_0(\theta)$   $r \rightarrow \infty$   
 $t \rightarrow \infty$

$$\boxed{A_0(\theta) \approx r^{1/2} p_{sc}(r, \theta, t) e^{-ikr} e^{i\omega t}} \quad (12.2)$$

This approx. can be easily improved to  $O(1/r^2)$ .

Also, Subst. (12.1) into Sommerfeld radiation cond. (5.5)

$$\frac{\partial p_{sc}}{\partial t} + c \frac{\partial p_{sc}}{\partial r} = \left[ -i\omega + \frac{c}{ik} - \frac{1}{2} r^{-1} \right] \frac{e^{ikr}}{r^{1/2}} A_0(\theta) e^{-i\omega t} + O(1/r^{3/2})$$

$c = \frac{\omega}{k}$

$$= O(1/r^{3/2})$$

This Sommerfeld radiation condition is not appropriate for numerical computation because is only  $O(1/r^{3/2})$ . Better radiation cond. (for numerical comp) have been defined by Bayliss, for example

$$\boxed{\frac{\partial p_{sc}}{\partial t} + c \frac{\partial p_{sc}}{\partial r} + c \frac{p_{sc}}{2r} = O(1/r^{5/2})}$$

(This problem) show this

For field amplitude from exact solution.

Exact solution:

$$p_{sc}(r, \theta, t_0) = \sum_{m=0}^{\infty} \frac{-\epsilon_m i^m J_m(kr_0)}{H_m^{(1)}(kr_0)} H_m^{(1)}(kr) \cos m\theta e^{-i\omega t}$$

$\epsilon_0=1, \epsilon_m=2, m \geq 1$

Using asymptotic expansion for  $H_m^{(1)}(kr) \xrightarrow{r \rightarrow \infty}$ .

$$p_{sc}(r, \theta, t_0) = \left(\frac{2}{k\pi r}\right)^{1/2} e^{i(kr - \pi/4)} P(\theta; kr_0) e^{-i\omega t_0}$$

where

$$P(\theta; kr_0) \equiv \sum_{m=0}^{\infty} \frac{-\epsilon_m J_m(kr_0)}{H_m^{(1)}(kr_0)} \cos m\theta.$$

Using Wilcox expansion, we notice that

$$\left(\frac{2}{k\pi}\right)^{1/2} e^{-i\pi/4} P(\theta; kr_0) = A_0(\theta)$$

$$\Rightarrow |A_0(\theta)| = \left(\frac{2}{k\pi}\right)^{1/2} |P(\theta; kr_0)|$$

$$\text{or } |P(\theta; kr_0)| = \left(\frac{k\pi}{2}\right)^{1/2} |A_0(\theta)| \rightarrow \text{Differential scatt. cross section normalized.}$$

Stop Criteria:

$$\max_{\substack{1 \leq i \leq N \\ 0 \leq j \leq M-1}} \left| \left| (p_{sc})_{i,j}^{n+1} \right| - \left| (p_{sc})_{i,j}^n \right| \right| < \epsilon$$

$\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$  are typical values for our experiments.

Stability Condition: for IVP.

In 2-D for Wave equation in Cartesian coords.

$$C \leq \left\| \left( \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right) \right\| = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t}$$

$$\Rightarrow \Delta t \leq \frac{1}{c} \sqrt{\Delta x^2 + \Delta y^2}$$

restriction on choice of  $\Delta t$ .

In polar coords.

$$C \leq \left\| \left( \frac{\Delta r}{\Delta t}, \frac{r \Delta \theta}{\Delta t} \right) \right\| = \frac{\sqrt{\Delta r^2 + r^2 \Delta \theta^2}}{\Delta t}$$

$$\Rightarrow \Delta t \leq \frac{1}{c} \sqrt{(\Delta r)^2 + r^2 \Delta \theta^2}$$

A Suffi. cond. is given by (More restrictive)

$$\Delta t < \frac{1}{c} \Delta r, \quad \Delta t < \frac{r}{c} \Delta \theta$$