3.4 Accuracy and stability.

Either 5-points or 9-points FDM for Poisson's equ.
Can be written as

$$A\vec{U} = \vec{F}.$$

with A given by (3.12) for 5-points

Leveque Book Notation

If $\Delta x = \Delta y$, 5-point is given by $\quad U_{i,j} \simeq U(x_i, x_j)$

$$\frac{1}{h^2}\left[U_{i,j-1} + U_{i-1,j} - 4U_{ij} + U_{i+1,j} + U_{i,j+1}\right] = f_{ij}$$

row-ordering

$$A \equiv \frac{1}{h^2}\begin{bmatrix} T & I & 0 & 0 & \cdots & 0 \\ I & T & I & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & \\ 0 & \cdots & & & I & T \end{bmatrix}, \quad T \equiv \begin{bmatrix} -4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -4 & 0 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & 1 & -4 \end{bmatrix}$$

(1.1)

It can be easily shown

$$LTE = \tau_{ij}(h) = \frac{1}{12} h^2 \left[U_{4x} + U_{4y}\right](x_i, y_j) + O(h^4).$$

Global error $\equiv U_{ij} - U(x_i, y_j) \equiv E^h_{ij}$

Again, $\quad A^h \vec{E}^h = -\vec{\tau}^h$

since $\quad A^h \vec{U} = \vec{F}, \quad A^h \vec{U}_{ex} = \vec{F} + \vec{\tau}^h$

Therefore, $\vec{E}^h = \|(A^h)^{-1}\| \, \|\vec{\tau}^h\|$

If $\|(A^h)^{-1}\| \le C$ indep. of h. $(h \le h_0)$ FDM is Stable and $\mathcal{O}(\vec{\tau}(h))$

FDM is convergent with rate of converg $= \mathcal{O}(h^2)$.

FDM is stable requires to find C s.t.

$$\|(A^h)^{-1}\|_{\substack{\text{some}\\\text{norm}}} \le C, \qquad h \le h_0$$

In $\|\cdot\|_\infty$ Complicated. Previous approach for 1-D with Green's fn. is much more complicated now.

It's easier in $\|\cdot\|_2$.

Back to <u>Section 2.10</u>

<u>Norm L_2 for matrices</u>

<u>Def.</u>- (spectral radius)

The S.R. of a matrix A

$$\rho(A) = \max|\lambda|, \quad \lambda \text{ eigenv. of } A.,$$

Def.- $\|\cdot\|_2$ for matrices

$$\|A\|_2 \equiv \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2$$

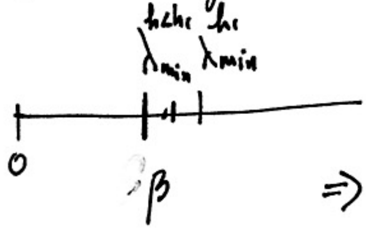
<u>thm.</u>- $A_{m\times m}$ matrix

i) $\|A\|_2 = [\rho(A^TA)]^{1/2}$

ii) If $A = A^T \Rightarrow \|A\|_2 = \rho(A) = \max_{1\le p\le m} |\lambda_p|$

iii) If $A = A^T$ and invertible $\Rightarrow \|A^{-1}\|_2 = \rho(A^{-1}) = \max_{1\le p\le m} \left|\frac{1}{\lambda_p}\right| = \left(\min_{1\le p\le m} |\lambda_p|\right)^{-1}$

To show stability, we need to find eigenvalues of A^h and show that they are bounded away from zero as $h \to 0$.



$\Rightarrow \quad \|(A^{-1})^h\| \le \frac{1}{\beta}, \quad h \le h_0$

Difficult to find eigenvalues in closed form for all A^h, $h \to 0$ in general. But, this case "relatively easy".

Eigenvalues are : $\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right), \quad p = 1, 2, .., m, \quad h = \frac{1}{m+1}$

Let's start with matrix A (2.10)

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & & \\ 0 & 1 & -2 & 1 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & & 1 & -2 \end{bmatrix} \qquad (3.1)$$

matrix (2.10) in book.

Eigenvectors are $\vec{u}^p$ with components, $u_j^p = \sin(p\pi j h)$, $p = 1.., m$, $j = 1.., m$

We also assume that $u_0^p = u_{m+1}^p = 0$

Checking: For $j=1,.., m$

$$(A\vec{u}^p)_j = \frac{1}{h^2}\left[u_{j-1}^p - 2u_j^p + u_{j+1}^p\right] =$$

$$= \frac{1}{h^2}\left[\sin(p\pi(j-1)h) - 2\sin(p\pi jh) + \sin(p\pi(j-1)h)\right]$$

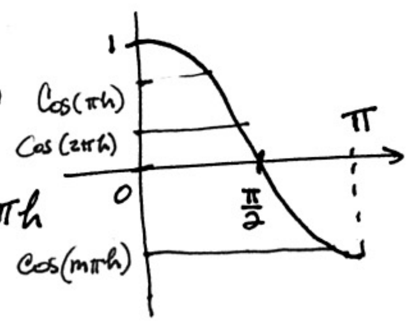
$$= \frac{2}{h^2}\left[\sin(p\pi jh)\cos(p\pi h) - \sin(p\pi jh)\right] =$$

$$= \frac{2}{h^2}\left[\cos(p\pi h) - 1\right]\sin(p\pi jh) = \lambda_p u_j^p.$$

Summarizing, we have

Eigenvalues: $\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$, $p = 1, 2, .., m$, $h = \frac{1}{m+1}$

Eigenvectors: $\vec{u}^p = \begin{pmatrix} \sin(p\pi h) \\ \sin(p\pi 2h) \\ \vdots \\ \sin(p\pi mh) \end{pmatrix}$



Since $h = \frac{1}{m+1}$ and $1 \le p \le m$

$$\pi h = \frac{1}{m+1}\pi \le p\pi h = p\pi\frac{1}{m+1} \le \frac{m}{m+1}\pi \le m\pi h$$

So $\lambda_1^h = \frac{2}{h^2}\left(\cos(\pi h) - 1\right)$ is the smallest eigenvalue in magnitude

And $\lambda_m^h = \frac{2}{h^2}\left(\cos(m\pi h) - 1\right)$ is the largest " " ".

Holds as $h \to 0$. $\lim_{h\to 0} \lambda_1^h = \frac{0}{0}$

$$\lambda_1^h = \frac{2}{h^2}\left(\cancel{1} - \frac{\pi^2 h^2}{2} + \frac{\pi^4 h^4}{4!} + \cdots - \cancel{1}\right) = -\pi^2 + O(h^2)$$

$$\lambda_m^h = \frac{2}{h^2}\left(\cos\left(\frac{m}{m+1}\pi\right) - 1\right) \xrightarrow[\substack{m\to\infty \\ h\to 0}]{} -\frac{4}{h^2} \to -\infty$$

Therefore, for any $h \le h_0$

$\lambda_1^h \simeq -\pi^2$ the smallest eigenvalue of (A^h).

Then, $\|(A^h)^{-1}\|_2 = \left(\min_{1\le p\le M} |\lambda_p|\right)^{-1} = \frac{2}{\pi^2}$ for $h \le h_0$.

As a result, the 2^{nd} order FDM for the DIR. BVP in sect 2.4 is stable and consequently the FDM is convergent with rate of convergence $\simeq O(h^2)$.

<u>Remark</u>: This approach represent an alternative for the one used in sect. 2.11 based on obtaining a bound for $\|(A^h)^{-1}\|$ using the $\|\cdot\|_\infty$ and Green's function.

For the matrix A given by (1.1) or (3.12) in book

$$\boxed{u_{ij}^{p,q} = \sin(p\pi i h)\sin(q\pi j h) \quad \text{eigenvects}}$$

$$\boxed{\lambda_{p,q} = \frac{2}{h^2}\left[(\cos(p\pi h)-1) + (\cos(q\pi h)-1)\right] \quad \text{Eigenvalues} \quad p,q = 1,2\ldots m}$$

Eigenvs. strictly negative $\Rightarrow$ A is negative definite

Eigenv. closest to the origin for any h

$$\boxed{\lambda_{1,1} = -2\pi^2 + \mathcal{O}(h^2)}$$

$$\Rightarrow \quad \|(A^h)^{-1}\| = \rho((A^{h})^{-1}) = \frac{1}{\lambda_{1,1}} \simeq + \frac{1}{2\pi^2} \Rightarrow \text{FDM is stable in } \|\cdot\|_2.$$

Comment on Cond. # $=$ $\boxed{K_2(A) = \|A\|_2 \|A^{-1}\|_2}$

Largest eigenv. of A $=$ $\boxed{\lambda_{m,m} \simeq \frac{-8}{h^2} \Rightarrow \|A^h\|_2 = \frac{8}{h^2}}$

$$\Rightarrow \boxed{K_2(A^h) \simeq \frac{4}{\pi^2 h^2} = \mathcal{O}\left(\frac{1}{h^2}\right) \xrightarrow[\text{as } h\to 0]{} \infty}$$

Very bad conditioned!