Theoretical Concepts.

Continuous IBVP : $\quad Lu(\vec{x}) = F, \; + BC + IC. \qquad (1)$

Finite diffce scheme : $\quad L_\Delta v_i^n = G_i^n \qquad (2)$

Defⁿ (Convergence)

A FDM (2) approximating a cont. init. bound. value prob. (1) is a convergent scheme at time t (fixed) $0 \le t \le T$ in a particular grid function norm $\|\cdot\|$ if

$$\|\vec{u}^n - \vec{v}^n\| \to 0, \quad \text{as} \quad \begin{matrix} n \to \infty; & \Delta t \to 0 \\ \Delta x \to 0; & n\Delta t \to t \le T \end{matrix}$$

Remark: Δx and Δt may not be independent as $\Delta t \to 0$ and $\Delta x \to 0$.

Consider the IBVP:

$$\begin{cases} u_t = \sigma u_{xx}, & 0 \le x \le 1, \; t > 0 & (3) \\ u(0,t) = \text{[illegible]}(t), & u(1,t) = \text{[illegible]}(t) & (4) \\ u(x,0) = f(x) & & (5) \end{cases}$$

and the numerical scheme FT-CS.

$$U_i^{n+1} = r\,U_{i-1}^n + (1-2r)\,U_i^n + r\,U_{i+1}^n, \quad i=1,.., m \qquad (6)$$

$$U_0^n = h(t_n) = h_n, \quad U_{m+1}^n = g(t_n) = g_n, \quad n=1,2\ldots T \qquad (7)$$

$$U_i^0 = f(x_i) = f_i, \qquad i=1,2\ldots m \qquad (8)$$

Thm.- The solution of the finite difference scheme (6)-(8) converges in the $\|\cdot\|_\infty$ to the solution of the IBVP (3)-(5) if

$$r \equiv \frac{\sigma \Delta t}{\Delta x^2} \le 1/2 .$$

Proof.- Substitution of the exact sln. $u(x,t)$ into (6)

$$u_i^{n+1} = r\,u_{i-1}^n + (1-2r)\,u_i^n + r\,u_{i+1}^n + \underbrace{+\,\Delta t\left[\frac{\Delta t}{2}(u_{tt})_i^{n+\theta} - \frac{\sigma \Delta x^2}{12}(u_{xxxx})_{i+\xi}^n\right]}_{= -\Delta t\, T_i^n} \qquad (9)$$

Subtracting (6) from (9)

$$\boxed{e_i^{n+1} = r\,e_{i-1}^n + (1-2r)\,e_i^n + r\,e_{i+1}^n - \Delta t\, T_i^n,} \qquad (10)$$

$i=1,2,..m$

Applying triangular inequality

$$|e_i^{n+1}| \le r\,|e_{i-1}^n| + |1-2r|\,|e_i^n| + r\,|e_{i+1}^n| + \Delta t\,|T_i^n|$$

since $r \le 1/2$

then
$$|e_i^{n+1}| \le r\,|e_{i-1}^n| + (1-2r)|e_i^n| + r\,|e_{i+1}^n| + \Delta t\,|T_i^n| \quad (11)$$

using $\|\cdot\|_\infty$
$$\le r\,\|\vec{e}^{\,n}\|_\infty + (1-2r)\|\vec{e}^{\,n}\|_\infty + r\|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{T}^n\|_\infty$$

$$= \|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{T}^n\|_\infty$$

$\therefore$
$$\|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{T}^n\|_\infty$$

Iterating over n.

$$\|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,n-1}\|_\infty + \Delta t\left(\|\vec{T}^n\|_\infty + \|\vec{T}^{n-1}\|_\infty\right)$$

$$\le \dots \le \underbrace{\|\vec{e}^{\,0}\|}_{\overset{IC}{=}0} + \Delta t\left(\|\vec{T}^n\|_\infty + \dots + \underbrace{\|\vec{T}^0\|_\infty}_{\overset{IC}{=}0}\right) \quad (12)$$

$$\Rightarrow \|\vec{e}^{\,n+1}\|_\infty \le n\Delta t\, T \le \underset{\text{finite \#}}{T}\, T.$$

Where $T \equiv \max\limits_{1\le j\le n} \|\vec{T}^j\|_\infty$

Defining

$$H \equiv \max_{\substack{0 \le t \le T \\ 0 \le x \le 1}} |u_{tt}| \quad \text{and} \quad P \equiv \max_{\substack{0 \le t \le T \\ 0 \le x \le 1}} |u_{xxxx}|$$

then
$$\|\vec{e}^{\,n+1}\|_\infty \le T \left[\frac{\Delta t}{2} H - \frac{\sigma \Delta x^2}{12} P \right] \xrightarrow[\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ n \to \infty}]{} 0$$

Remarks: 1) Δt and Δx both tend to zero and also satisfy the condition $r \le 1/2$ or $\boxed{\Delta t \le \frac{1}{2\sigma} (\Delta x)^2}$.

2) The inequality (11) is different for $i=1$ and $i=m$.

In fact,
$$|e_1^{n+1}| \le r\,|\underbrace{e_0^n}_{=0}| + (1-2r)\,|e_1^n| + r\,|e_2^n| + \Delta t\,|\tau_i^n| \quad (13)$$

$$|e_m^{n+1}| \le r\,|e_{m-1}^n| + (1-2r)\,|e_m^n| + r\,|\underbrace{e_{m+1}^n}_{=0}| \quad (14)$$

where $|e_0^n| = 0$ and $|e_{m+1}^n| = 0$ assuming there is not round-off errors in the comput. of the BCs.

In both cases

(13) and (14) $\Rightarrow$ $|e_i^{n+1}| \le (1-r)\|\vec{e}^{\,n}\|_\infty + \Delta t \|\vec{\tau}^{\,n}\|_\infty$

for $i=1$ or m $\quad \le \|\vec{e}^{\,n}\| + \Delta t \|\vec{\tau}^{\,n}\|_\infty$

The FDM studied so far can be represented by a linear system as

$$\boxed{\vec{U}^{n+1} = L_\Delta \vec{U}^n + \vec{f}^n.} \tag{5.1}$$

I) For instance, FT-CS. for IBVP (3)-(5).

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_m^{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-2r & r & 0 & 0 & \cdots & 0 \\ r & 1-2r & r & 0 & \cdots & 0 \\ 0 & r & 1-2r & r & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ & & r & 1-2r & r & \\ 0 & \cdots & 0 & & r & 1-2r \end{bmatrix}}_{\equiv L_\Delta} \begin{bmatrix} U_1^n \\ \vdots \\ U_i^n \\ \vdots \\ U_m^n \end{bmatrix} + \underbrace{\begin{bmatrix} +r h_0 \\ 0 \\ \vdots \\ 0 \\ r g_{m+1} \end{bmatrix}}_{\equiv \vec{f}^n}$$

II) <u>Crank-Nicholson</u>

$$A^h \vec{U}^{n+1} = B^h \vec{U}^n + \vec{C}^n \Rightarrow \vec{U}^{n+1} = (A^h)^{-1} B^h \vec{U}^n + (A^h)^{-1} \vec{C}^n.$$

or $$\vec{U}^{n+1} = L_\Delta \vec{U}^n + \vec{f}^n.$$

where $L_\Delta \equiv (A^h)^{-1} B^h$ and $\vec{f}^n \equiv (A^h)^{-1} \vec{C}^n$.

Consider

1) Homogeneous initial value problems (continuous)

2) Same FDM apply to two different IVP.

$$\begin{cases} \vec{U}^{n+1} = L_\Delta \vec{U}^n, & \vec{U}^0 = \phi & (6.1) \\ \vec{V}^{n+1} = L_\Delta \vec{V}^n, & \vec{V}^0 = \psi & (6.2). \end{cases}$$

Def. – A finite diff. scheme (6.1) is stable

there exists $C > 0$, such that

$$\|\vec{U}^n - V^n\| \le C \|\vec{U}^0 - V^0\|, \qquad \begin{matrix} \Delta x \to 0 \\ \Delta t \to 0 \\ n\Delta t \le T \\ n \to \infty \end{matrix}$$

Remark: 1) C is uniform does not depend on Δx, Δt, $\vec{U}^0$, or V^0.

2) C may be greater than 1.