Thm.- If $J(\xi,\eta) \neq 0$, for all $(\xi,\eta) \in \mathcal{D}'$

Show that the inverse of the Laplace system of equations

$$\begin{cases} \xi_{xx} + \xi_{yy} = 0, \\ \eta_{xx} + \eta_{yy} = 0, \end{cases} \qquad (x,y) \in \mathcal{D} \qquad (1)$$

Can be written as Winslow's quasilinear elliptic System,

$$\begin{cases} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0, \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0, \end{cases} \qquad (\xi,\eta) \in \mathcal{D}' \ (2)$$

where $\alpha \equiv x_\eta^2 + y_\eta^2$, $\beta \equiv x_\eta x_\xi + y_\xi y_\eta$, $\gamma = x_\xi^2 + y_\xi^2$

Proof.- Consider the transformations

$$T: \mathcal{D}' \longrightarrow \mathcal{D}$$
$$(\xi,\eta) \rightarrow (x(\xi,\eta), y(\xi,\eta))$$

and

$$T^{-1}: \mathcal{D} \longrightarrow \mathcal{D}'$$
$$(x,y) \longrightarrow (\xi(x,y), \eta(x,y))$$

Thus, $$x = x(\xi,\eta)\,, \qquad y = y(\xi,\eta)$$

or
$$x - x(\xi,\eta) = 0\,, \qquad y - y(\xi,\eta) = 0$$

or
$$\begin{cases} x - x\big(\xi(x,y), \eta(x,y)\big) = 0 & (2.1) \\ y - y\big(\xi(x,y), \eta(x,y)\big) = 0 & (2.2) \end{cases}$$

Differentiating (2.1)-(2.2) w.r. to x

$$1 - x_\xi\big(\xi(x,y), \eta(x,y)\big)\,\xi_x - x_\eta\big(\xi(\,), \eta(\,)\big)\,\eta_x = 0 \quad (2.3)$$
$$0 - y_\xi\big(\xi(\,), \eta(\,)\big)\,\xi_x - y_\eta\big(\xi(\,), \eta(\,)\big)\,\eta_x = 0 \quad (2.4)$$

In matrix form
$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (2.5)$$

Analogously, differentiating (2.1)-(2.2) w.r.t. y

$$\begin{cases} 0 - x_\xi(\,)\,\xi_y - x_\eta(\,)\,\eta_y = 0 \\ 1 - y_\xi(\,)\,\xi_y - y_\eta(\,)\,\eta_y = 0 \end{cases}$$

In matrix form
$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_y \\ \eta_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.6)$$

Then, Combining (2.3) and (2.4)

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.1)$$

Therefore,

$$\text{if } \mathcal{J} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \text{ then } \mathcal{J}^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

but

$$\mathcal{J}^{-1} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}, \quad \text{Since } J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0.$$

As a consequence,

$$\frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

and

$$\boxed{\begin{aligned} \xi_x &= \frac{y_\eta}{J}, & \xi_y &= -\frac{x_\eta}{J} \\ \eta_x &= \frac{-y_\xi}{J}, & \eta_y &= \frac{x_\xi}{J}. \end{aligned}} \qquad (3.2)$$

First order derivatives of $\xi(x,y)$ and $\eta(x,y)$ in terms of first order derivatives of $x(\xi,\eta)$ and $y(\xi,\eta)$.

Now, we want to obtain Second order derivatives of $\xi(x,y)$ and $\eta(x,y)$ in terms of Second order and first order derivatives of $x(\xi,\eta)$ and $y(\xi,\eta)$.

Differentiating (2.3) and (2.4) again w.r.t. x

$$\begin{cases} x_{\xi\xi}\,\xi_x^2 + x_{\xi\eta}\,\xi_x\eta_x + x_\xi\,\xi_{xx} + x_{\eta\xi}\,\eta_x\xi_x + x_{\eta\eta}\,\eta_x^2 + x_\eta\,\eta_{xx} = 0 \\ y_{\xi\xi}\,\xi_x^2 + 2y_{\xi\eta}\,\xi_x\eta_x + y_\xi\,\xi_{xx} + {}_{\eta\eta}\,\eta_x^2 + y_\eta\,\eta_{xx} = 0 \end{cases}$$

From these equations the following system for ξ_{xx} and η_{xx} is obtained

$$x_\xi\,\xi_{xx} + x_\eta\,\eta_{xx} = -\left(x_{\xi\xi}\,\xi_x^2 + 2x_{\xi\eta}\,\xi_x\eta_x + x_{\eta\eta}\,\eta_x^2\right)$$

$$\overset{\text{using (3.2}}{=} -\frac{1}{J^2}\left(x_{\xi\xi}\,y_\eta^2 - 2x_{\xi\eta}\,y_\xi y_\eta + x_{\eta\eta}\,y_\xi^2\right) \quad (4.1)$$

$$y_\xi\,\xi_{xx} + y_\eta\,\eta_{xx} \overset{\text{using (3.2)}}{=} -\frac{1}{J^2}\left(y_{\xi\xi}\,y_\eta^2 - 2y_{\xi\eta}\,y_\xi y_\eta + y_{\eta\eta}\,y_\xi^2\right) \quad (4.2)$$

Let's define

$$A = -\left(x_{\xi\xi}\,y_\eta^2 - 2x_{\xi\eta}\,y_\xi y_\eta + x_{\eta\eta}\,y_\xi^2\right)$$

$$B = -\left(y_{\xi\xi}\,y_\eta^2 - 2y_{\xi\eta}\,y_\xi y_\eta + y_{\eta\eta}\,y_\xi^2\right)$$

Thus, (4.1) and (4.2) can be written as

$$\begin{cases} x_\xi \xi_{xx} + x_\eta \eta_{xx} = \dfrac{1}{J^2} A & (5.1) \\ y_\xi \xi_{xx} + y_\eta \eta_{xx} = \dfrac{1}{J^2} B & (5.2) \end{cases}$$

Proceeding in an analogous form, we can differentiate (2.3) and (2.4) w.r.t. "y", and obtain the system

$$\begin{cases} x_\xi \xi_{yy} + x_\eta \eta_{yy} = \dfrac{1}{J^2} C & (5.3) \\ y_\xi \xi_{yy} + y_\eta \eta_{yy} = \dfrac{1}{J^2} D & (5.4) \end{cases}$$

where

$$C \equiv -\left(x_{\xi\xi} x_\eta^2 - 2 x_{\xi\eta} x_\xi x_\eta + x_{\eta\eta} x_\xi^2 \right)$$

$$D \equiv -\left(y_{\xi\xi} x_\eta^2 - 2 y_{\xi\eta} x_\xi x_\eta + y_{\eta\eta} x_\xi^2 \right)$$

From here, there is only two more steps to prove the theorem.

1) a) Solve System (5.1)-(5.2) for ξ_{xx} and η_{xx}.

   b) Solve System (5.3)-(5.4) for ξ_{yy} and η_{yy}.

2) Substitute $\xi_{xx}, \eta_{xx}, \xi_{yy}, \eta_{yy}$ obtained in (1) into the system

$$\begin{cases} \xi_{xx} + \xi_{yy} = 0 \\ \eta_{xx} + \eta_{yy} = 0 \end{cases}$$

and use that $J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0$ to obtain the Winslow system of equations.

$$\begin{cases} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \end{cases}$$

$$\alpha \equiv x_\eta^2 + y_\eta^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi^2 + y_\xi^2 .$$