Math 511-01

Final Project (Numerical Part)

Name: ____________________________

Due before 6 pm, XXday April XX, 2009

Answer all questions and show all your work carefully. This is an individual test. You should not discuss the questions and your answers with any one. *BYU students should seek to be totally honest in their dealings with others. They should complete their own work and be evaluated based upon that work. (Honor Code)*

Prof. Vianey Villamizar

Problem No.	**Points**
1)	
2)	
3)	
4)	
Total	

Formulation

This Final Project consists of numerically solving the BVP modelling the **two-dimensional acoustic scattering of a plane pressure wave p_{inc} from an empty, infinite, cylindrical cavity of arbitrary cross section** given by

$$\nabla^2 p_{sc} + k^2 p_{sc} = 0 \quad \text{in } \mathcal{D} \tag{1}$$

$$p_{sc} = -p_{inc} \quad \text{on obstacle boundary } \mathcal{C} \tag{2}$$

$$\sqrt{r}\left[(p_{sc})_r - ik(p_{sc})\right] \to 0 \quad \text{as} \quad r \to \infty, \tag{3}$$

where

$$p_{inc} = e^{ik\left(x\cos\delta + y\sin\delta\right)} \quad \text{in } \mathcal{D}. \tag{4}$$

For computational purposes the radiation condition needs to be replaced by an absorbing condition at an artificial infinite boundary (a circle of radius R_∞). Then, use the following condition instead of (3),

$$\frac{\partial p_{sc}}{\partial r} - ikp_{sc} + \frac{p_{sc}}{2r} = 0, \qquad \text{on artificial boundary} \quad r = R_\infty \tag{5}$$

The obstacles have circular and astroid cross sections such as

$$\textbf{Astroid:} \qquad x(\theta) = \frac{1}{2}(3\cos(\theta) + \cos(3\theta)), \quad \text{and} \tag{6}$$

$$y(\theta) = \frac{1}{2}(3\sin(\theta) - \sin(3\theta)), \qquad 0 \le \theta \le 2\pi \tag{7}$$

$$\textbf{Circle}: \qquad x(\theta) = 2\cos(\theta), \quad y(\theta) = 2\sin(\theta), \qquad 0 \le \theta \le 2\pi \tag{8}$$

Parametric equation for the artificial boundary $\mathcal{B}$

$$x(\theta) = R_\infty \cos(\theta), \quad y(\theta) = R_\infty \sin(\theta), \qquad 0 \le \theta \le 2\pi \tag{9}$$

In order to apply the numerical method to arbitrary shaped obstacles, the BVP (1)-(3) is reformulated in curvilinear coordinates ξ and η (Winslow-type) as follows

$$\frac{1}{J^2}\Big(\alpha(p_{sc})_{\xi\xi} - 2\beta(p_{sc})_{\xi\eta} + \gamma(p_{sc})_{\eta\eta}\Big) + k^2 p_{sc} = 0, \quad \text{for} \quad (\xi,\eta), \in \mathcal{D}', \tag{10}$$

$$p_{sc}(\xi,\eta) = -p_{inc}(\xi,\eta), \quad \text{at} \quad \eta = \eta_1, \tag{11}$$

$$\frac{1}{R_\infty J}\Big((xy_\eta - yx_\eta)(p_{sc})_\xi + (yx_\xi - xy_\xi)(p_{sc})_\eta\Big) - ikp_{sc} + \frac{p_{sc}}{2R_\infty} = 0, \quad \text{at } \eta = \eta_{N_2} \tag{12}$$

$$p_{inc}(\xi,\eta) = e^{ik\left(x(\xi,\eta)\cos\delta + y(\xi,\eta)\sin\delta\right)} \quad (\xi,\eta) \in \mathcal{D}'. \tag{13}$$

This is the BVP, defined on the annular region bounded by the obstacle bounding curve and by the outer circle of radius R_∞, that you are asked to solve numerically using centered finite-difference.

Questions

As part of this project you will need to answer the following questions:

1. Show that if $J(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \mathcal{D}'$ (computational domain) then, Laplace equation in generalized curvilinear coordinates can be written as

$$\nabla^2_{xy} f = \nabla^2_{\xi\eta} \tilde{f} = \frac{1}{J^2}\left(\alpha \tilde{f}_{\xi\xi} - 2\beta \tilde{f}_{\xi\eta} + \gamma \tilde{f}_{\eta\eta}\right) \tag{14}$$

$$+\frac{1}{J^3}\left(\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta}\right)\left(y_\xi \tilde{f}_\eta - y_\eta \tilde{f}_\xi\right) \tag{15}$$

$$+\frac{1}{J^3}\left(\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta}\right)\left(x_\eta \tilde{f}_\xi - x_\xi \tilde{f}_\eta\right), \tag{16}$$

   where $\alpha = x_\eta^2 + y_\eta^2$, $\beta = x_\eta x_\xi + y_\xi y_\eta$, $\gamma = x_\xi^2 + y_\xi^2$, and $J(\xi,\eta)$ represents the jacobian of the transformation T: $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$ from a rectangular computational domain $\mathcal{D}'$ with coordinates ξ and η into the two-dimensional truncated physical domain with coordinates x and y.

2. Show that if $J(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \mathcal{D}'$ then, the absorbing boundary condition (5) can be written in terms of generalized curvilinear coordinates such as in equation (12).

3. Use centered finite-difference to transform BVP (10)-(12) into an algebraic linear system of equations whose unknowns are the value of p_{sc} at the grid points. **Clearly write this linear system separated from the code so I can grade it.** More precisely, define the sparse matrix A completely. Define the unknown vector $\mathbf{U}$, and the forcing vector $\mathbf{b}$ with all details. This exercise is necessary before you start coding your algorithm.

4. Execute your code using the following values for the parameters: Wave number $k = 2\pi$, Angle of incidence $\delta = 0$, Artificial radius $R_\infty = 6$, Grid size $N_1 \times N_2 = 181 \times 101$, Grid tolerance $Tol_g = 1e-5$.

   Then, make a report of your results by:

   (a) Creating a table with the following columns:
   Shape – $\min_{1\leq i\leq N_1}(|p_{sc}(\xi_i, N_2)|)$ – $\max_{1\leq i\leq N_1}(|p_{sc}(\xi_i, N_2)|)$.

   (b) Make a contour graph with 10 levels ("contourf(x,y,abs(p_{total}),10)") showing the amplitude of the harmonic steady state of the total pressure field $|p_{total}(\xi_i,\eta_j)|$ given by $p_{total} = p_{sc} + p_{inc}$. Also, use the MATLAB surface drawer command "surf()" to obtain another view of the the total pressure field.

   (c) Make a polar plot of the amplitude of the scattered field at $\eta = N_2$. That is, plot $|p_{sc}(\xi_i, N_2)|$ in terms of the polar angle θ varying on $[0, 2\pi]$. Find the angles where the maximum and minimum are reached. Do this for each one of the shapes.

(d) For the circular case, an analytical exact solution p_{exact} can be obtained for the scattered pressure p_{sc}. This is given by

$$p_{exact}(r, \theta, t) = \sum_{m=0}^{\infty} B_m H_m^{(1)}(kr) \cos m\theta \tag{17}$$

where $B_m = \dfrac{-\epsilon_m i^m J_m(kr_0)}{H_m^{(1)}(kr_0)}$, $\epsilon_0 = 1$, $\epsilon_m = 2$ (for $m \geq 1$), J_m is the Bessel function of the first kind of order m, $H_m^{(1)}$ represents a Hankel function of order m, and r_0 is the circular cylinder radius. Use this exact solution to compute the point-wise error:

$$Error_i = abs\Big(abs(p_{sc}(\xi_i, N_2)) - abs(p_{exact}(\xi_i, N_2))\Big), \quad i = 1 \ldots N_1.$$

Use only 20 terms of the infinite series (17) Make sure that the comparison is made on the same grid points at the artificial infinite boundary. Then, report: $max(Error_i)$ and $Average(Error_i)$. For p_{exact} function, you will need to use Bessel and Hankel functions in your code, MATLAB has these functions built into his kernel. In fact, you can use $besselh(m, kr_0)$ for Hankel functions and $besselj(m, kr_0)$ for Bessel functions.

Final Recommendations

i) Read carefully all the items and try to give answer to all of them.

ii) I have carefully worked out all the parts of this exam myself. However, I may have overlooked something in the proposed problems, if you think so modify the question and provide an answer to the modified question. In such event, clearly document the error. Points will be deducted if the question in its original form is correct.

iii) I think this test is reasonable and doable in a week. However, you will need TO START EARLY. I will be more than willing to clarify questions as you go along.

iv) THIS IS AN INDIVIDUAL PROJECT! You can use any resource available to you except your classmates. Make an effort to write your own algorithm and obtain your very own answer. Two individuals very rarely write the same code (programming is very personal).