Math 511
Section 1

Computational Part
Midterm Exam 1

Winter Semester 2009

Name: ____________________________

Due Friday March 13 in Class

Answer all questions and show all your work carefully. Make frequent comments on your codes so that I can understand your algorithm. Also, present detailed results to answer the questions in each problem.

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Problem No.	**Points**
1.-)	
2.-)	
Total	

1. Consider the nonhomogeneous Helmholtz equation in 2-D

$$\nabla^2_{x,y} u + k^2 u = u_{xx} + u_{yy} + k^2 u = f(x, y). \tag{1}$$

   (a) Write Equation (1) (with $f(x, y) = 0$) in terms of polar coordinates. Show that under radial symmetry, it is possible to reduce Helmholtz equation to Bessel's equation of order zero

$$z^2 \frac{d^2 \hat{u}}{dz^2} + z \frac{d\hat{u}}{dz} + z^2 \hat{u} = 0,$$

   Hint: Introduce the change of variables $z = kr$, where $r = \sqrt{x^2 + y^2}$.

   (b) Set up a Dirichlet boundary value problem (BVP) for Equation (1) defined on the square domain: $D = [0, 1] \times [0, 1]$, such that $u(x, y) = j_0(k\sqrt{x^2 + y^2}) + \cos(x) + \sin(y)$ is the solution. The function $j_0(kr) = j_0(k\sqrt{x^2 + y^2})$ is the Bessel's function of order zero.

   (c) The *5-point stencil* finite difference scheme for Equation (1) is defined by

$$\nabla^2_5 u_{ij} + k^2 u_{ij} = \frac{1}{h^2} (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}) + k^2 u_{ij} = f_{ij}, \tag{2}$$

   where $u_{ij} \approx u(x_i, y_j)$. What is the order of the leading term of the local truncation error?

      i. (Redoing an item of theoretical part of Midterm 1) Show that the local truncation error of the *5-point stencil* can be written as

$$(\tau_5)_{ij} = \frac{1}{12} h^2 \left[\nabla^2 f - k^2 f + k^4 u - 2 \left(u_{xx}\right)_{yy} \right]_{ij} + \mathcal{O}(h^4). \tag{3}$$

      ii. Use (3) to implement a fourth order *deferred corrections* (DC) method for (1). Show (analytically) that the new finite difference scheme is 4th order instead of second order.

   (d) Write and test a DC code to solve the BVP defined in part (b). Assume $k = 20$ and that $\Delta x = \Delta y = h$. By running appropriate experiments, verify the fourth order convergence of the numerical solution to the exact solution. List a table showing the order of the error for the different experiments and find an average order p (slope) using least squares, as done in the homework.

2. Define a symmetric 9-point stencil numerical method for equation (1).

   (a) Determine the truncation error in terms of f and u only, in a similar way as done in part (c)(i) of Question (1). However, for the 9-point stencil, there are not derivatives involved for the leading term of the truncation error.

   (b) Use part (2a) to construct a 9-point fourth order numerical method for the BVP defined in Question (1) (b). Show (analytically) that the new finite difference scheme is actually 4th order. Write and test your code to solve the BVP defined in Question (1) part (b) assuming $k = 20$ and $\Delta x = \Delta y = h$. By running appropriate experiments, verify the fourth order convergence of the numerical solution to the exact solution. List a table showing the order of the error for the different experiments and find an average order p (slope) using least squares, as done in the homework.