# Summary : Conservation Laws

$\mathcal{U}(\vec{x},t)$: Density of physical variable. $\left\{\begin{array}{l}\text{mass}\\ \text{energy}\\ \text{population}\end{array}\right\}$

PV: Physical variable.

APV: Amount of physical variable.

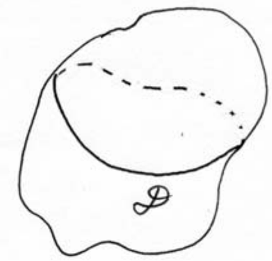
$\vec{\phi}(\vec{x},t)$: Flux of physical variable. , $\mathcal{D}$: Closed Volume bounded by Surface $S$.

<u>Definition</u>:- Flux of a physical variable is the amount of PV flowing through the domain bounding surface per unit of area and per unit of time.

## Physical Units.

$$[\mathcal{U}] = \frac{APV}{L^3}, \qquad [\vec{\phi}] = \frac{APV}{L^2 * t}$$

## Conservation Law:




Domain $\mathcal{D}$.
bounded by surface $S$.

| Rate of Change of the total APV inside $\mathcal{D}$ | = | Net rate of flow accross surface $S$ | + | Rate at which APV is introduced or taking out from domain $\mathcal{D}$. |
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## Mathematical representation

$$\frac{d}{dt}\int_{\mathcal{D}} u(\vec{x},t)\,dv = -\int_{S} \vec{\phi}(\vec{x},t)\cdot\hat{n}(\hat{x})\,ds + \int_{\mathcal{D}} f(\vec{x},t)\,dv \qquad (2.1)$$

Integral representation of the conservation law after applying the divergence theorem.

$$\int_{\mathcal{D}} \left[ \frac{\partial u}{\partial t}(\vec{x},t) + \nabla\cdot\vec{\phi} - f(\vec{x},t)\right] dv = 0 \qquad (2.2)$$

To transform (2.1) into (2.2), we need the functions $u$ and $\vec{\phi}$ to be Sufficiently Smooth. For example, we may require $u$ and $\vec{\phi}$ to be $C^1[\Omega \times T]$.

# PDE obtained from integral representation.

$$\boxed{\frac{\partial u}{\partial t}(\vec{x},t) + \nabla \cdot \vec{\phi}(\vec{x},t) = f(\vec{x},t)}, \quad \begin{array}{l}\vec{x} \in \Omega \\ t > 0.\end{array} \tag{3.1}$$

## I) Conservation of mass:

$u(\vec{x},t) = \rho(\vec{x},t)$ : density.

$\vec{\phi}(\vec{x},t)$ : mass flux $\quad \frac{Mass}{L^2 * t}$

Physically, mass is transported by particles of fluid flowing through the bounding surface $S$ enclosing the volume $D$. This process is called Convection.

Notice that $\quad [\rho \vec{v}] = \frac{M}{L^3} \frac{L}{t} = \frac{M}{L^2 * t}$

Therefore, a good definition for the flux vector is given by

$$\vec{\phi}(\vec{x},t) \equiv \rho \vec{v}.$$

Substitution into (2.1) leads to

$$\frac{d}{dt}\int_{\mathcal{D}} \rho(\hat{x},t)\,dv = -\int_S \rho(\hat{v}\cdot\hat{n})\,ds + \int_{\mathcal{D}} f(\hat{x},t)\,dv \qquad (4.1)$$

$\int_S \rho(\vec{v}\cdot\hat{n})\,ds$ is called Convective Surface term.

If there are not sources of mass inside $\mathcal{D}$ then

$$f(x,t) \equiv 0.$$

and (4.1) reduces to

$$\frac{d}{dt}\int_{\mathcal{D}} \rho(\vec{x},t)\,dv = -\int_S \rho(\vec{v}\cdot\hat{n})\,ds$$

Assuming $\rho(\hat{x},t)$ and $\vec{v}(\hat{x},t)$ are $C^1(\Omega)$. $(\mathcal{D}\subset\Omega)$

we obtain the PDE

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0} \quad \text{Continuity equation.}$$

(4.1)

If flow is steady and incompressible

$\rho(\vec{x},t) \equiv$ Const.

and equ. (4.1) reduces to

$$\boxed{\nabla \cdot \vec{v} = 0}$$

## II) <u>Heat Equation</u>. <u>Diffusion equations.</u>

Replacing $u(\vec{x},t) = \rho C T(\vec{x},t)$

$\rho$: density, $C$: specific heat at constant Vol.

$T$: temperature at $\vec{x}$ at time $t$.

and $\vec{\phi}(\vec{x},t) = -K_0 \nabla T$ Constitutive Law known as Fourier's Law of heat conduction.

into (3.1)

$$\rho c \frac{\partial T}{\partial t} = -\nabla \cdot (-K_0 \nabla T) = K_0 \nabla^2 T$$

where $f(\vec{x},t) \equiv 0$, and $\rho$, $c$, and $K_0$ are constants properties.

or $$\boxed{\frac{\partial T}{\partial t} = \kappa \nabla^2 T,} \quad \text{where } \kappa \equiv \frac{K_0}{\rho c} \quad (5.1)$$

This is called heat conduction equation

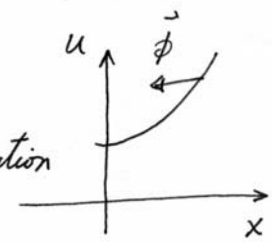
If we consider a chemical mixture

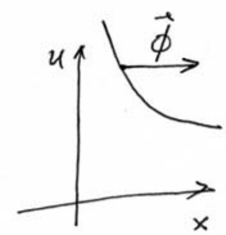
$u(\vec{x},t)$: density or concentration

$$\boxed{\vec{\phi}(\vec{x},t) = -D\,\nabla u}$$ Fick's law

Similar to Fourier's law

particles move from high concentration to low concentrations.




Equation (3.1) for $D$ constant and $f \equiv 0$ reduces to

$$\boxed{u_t = D\,\nabla^2 u}$$

$D$ is called diffusivity $[D] = \frac{L^2}{t}$.

## One dimensional heat conduction IBVP. $\Omega = [0, L]$.

In order to Solve (5.1), we need to impose an initial boundary condition (I.C) and two boundary conditions. (BC's)

$$\text{I.C.}: \quad T(x,0) = f(x), \qquad x \in (0, L).$$

Boundary conditions possibilities are

Dirichlet conditions:

$$T(0,t) = A(t)$$
$$T(L,t) = B(t)$$

Neumann Conditions:

$$\frac{\partial T}{\partial x}(0,t) = C(t)$$
$$\frac{\partial T}{\partial x}(L,t) = D(t)$$

Robin Conditions:

$$\left(\alpha_1 \frac{\partial T}{\partial x} + \beta_1 T\right)(0,t) = H(t)$$

$$\left(\alpha_2 \frac{\partial T}{\partial x} + \beta_2 T\right)(L,t) = G(t)$$

or any combination of them.