Summarizing <u>Example 1.</u>

<u>Unsteady heat conduction in a rod.</u>

<u>Assumptions:</u> i) Rod is laterally perfectly insulated.

ii) Rod is uniformly heated. thermal energy density varies only from one cross section to another.

$v(x,t)$: Temperature at cross section located in x at time t.

$e(x,t)$: Thermal energy density.

$$e(x,t) = c(x)\,\rho(x)\,v(x,t)$$

$Q(x,t)$: Heat energy per unit of volume, generated or taken away inside the slice

$\phi(x,t)$: Heat flux. Flow of energy per unit of time and per unit of surface area.

Principle of conservation of energy leads to

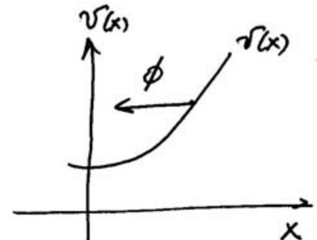
$$\frac{\partial}{\partial t}\left(e(\bar{x},t)\, A\, \Delta x\right) = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(\bar{x},t)A\Delta x$$

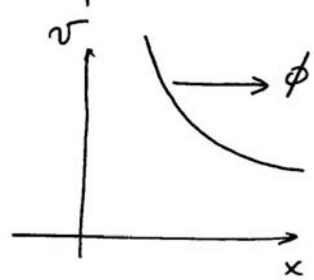
Dividing by $A\Delta x$ and taking $\lim_{\Delta x \to 0}$, we arrive to

$$\boxed{\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)} \qquad (1)$$

Fourier's law of heat conduction:

$$\phi(x,t) = -K_0 \frac{\partial v}{\partial x}$$





Substitution in (1) leads to

$$c\rho \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial v}{\partial x}\right) + Q(x,t)$$

If $K_0(x) \equiv$ constant
↳ Thermal conductivity

then

$$\boxed{\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + \frac{Q}{c\rho}} \qquad (2), \quad 0 < x < L,\ t > 0$$

→ It is known.
→ they are given.

If No Sources or Sinks : $Q = 0$.

$$\boxed{\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}} \quad 0 < x < L,\ t > 0 \qquad (3)$$

k : Thermal diffusivity.

(3) is called equation of heat conduction. It is of "parabolic type". It also models a diffusion process.

In order to solve (2) or (3), we need to impose two boundary conditions (BC's) and one initial condition (I.C.).

For example,

$$\begin{cases} v(0,t) = A(t) \\ v(L,t) = B(t) \end{cases} \quad \text{Dirichlet conditions.}$$

or

$$\begin{cases} \dfrac{\partial v}{\partial x}(0,t) = C(t) \\ \dfrac{\partial v}{\partial x}(L,t) = D(t) \end{cases} \quad \text{Neumann condition}$$

or

$$\begin{cases} \left[\alpha_1 \dfrac{\partial v}{\partial x} + \beta_1 v\right](0,t) = H(t) \\ \left[\alpha_2 \dfrac{\partial v}{\partial x} + \beta_2 v\right](L,t) = G(t) \end{cases} \quad \text{Robin conditions}$$

or any combination of them. Discuss physical meaning!

Plus an initial Condition:

$$v(x,0) = f(x), \quad 0<x<L.$$

The set of equations:

$$\begin{cases} \dfrac{\partial v}{\partial t} = k \dfrac{\partial^2 v}{\partial x^2} + \dfrac{Q}{c\rho}, & \begin{array}{l} 0<x<L \\ t>0 \end{array} \\ v(0,t) = A, & t>0 \\ \dfrac{\partial v}{\partial x}(L,t) = 0, & t>0 \\ v(x,0) = f(x), & 0<x<L \end{cases}$$

form an initial boundary Value problem (IBVP).

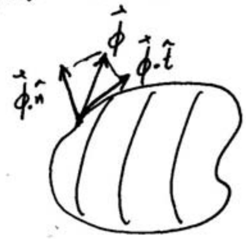
there are several analytic techniques to obtain its solution

In 2-D or 3-D, Consider an arbitrary region R

Want to study conservation of heat energy inside Region R bounded by surface S.

$$\text{Heat energy inside } R = \iiint_R c(x,y,z)\,\rho(x,y,z)\,V(x,y,z,t)\,dv$$

ϕ or heat flux is now a vector $\vec{\phi}(x,y,z,t)$.



$$\text{Flux of heat energy out of region } R = \iint_S \vec{\phi}(x,y,z,t)\cdot\hat{n}(x,y,z)\,ds$$

$$\overset{\text{Div thm}}{=} \iiint_R \nabla\cdot\phi(x,y,z,t)\,dv$$

Heat energy per unit of Vol. inside $R = Q(x,y,z,t)$.

Therefore, Conservation of energy inside R leads to

$$\frac{\partial}{\partial t}\iiint_R c\rho v\,dv \overset{\text{ASK ?}}{=} -\iint_S \vec{\phi}\cdot\hat{n}\,ds + \iiint_R Q(\vec{x},t)\,dv$$

$$= -\iiint_R \nabla\cdot\vec{\phi}\,dv + \iiint_R Q\,dv$$

or

$$\iiint_R \left[c\rho \frac{\partial v}{\partial t} + \nabla \cdot \phi - Q \right] dv = 0$$

True for any region R, then

$$c\rho \frac{\partial v}{\partial t} = -\nabla \cdot \phi + Q$$

Again, Fourier's Law of Heat conduction:

$$\phi = -K_0 \nabla v$$

$$\Rightarrow \quad \boxed{c\rho \frac{\partial v}{\partial t} = \nabla \cdot (-K_0 \nabla v) + Q.} \qquad (2.1)$$

If thermal conductivity K_0 is constant and $Q = 0$. (no sources of Heat energy)

$$\frac{\partial v}{\partial t} = k \, \nabla \cdot (\nabla v)$$

then $\boxed{\frac{\partial v}{\partial t} = k \nabla^2 v}$ Heat or diffusion equ. (2.2)

Possible BC's.: $v(x,y,z,t) = T(x,y,z,t)$ on boundary S.

$$\frac{\partial v}{\partial n} = \nabla v \cdot \hat{n}\,(x,y,z,t) = H(x,y,z,t) = \begin{cases} \neq 0 \\ = 0, \text{ insulated.} \end{cases}$$

IC: $v(x,y,z,0) = f(x,y,z)$. → Equations of elliptic type.

Steady state + No heat sources + Constant thermal properties in (2.1) leads to

$$\boxed{\begin{array}{ll} \nabla^2 v = -\dfrac{Q}{K_0} & \text{Poisson's equ.} \\ \nabla^2 v = 0, & \text{Laplace's equation} \end{array}}$$

If $Q = 0$

Discuss analytic techniques and limitations. !

Well - Posedness.-

- Existence of solution
- Uniqueeness of solution
- stability. Continuous dependence on initial and boundary conditions data.

Theorem.- (Uniqueness)

The BVP

$$v_t - K v_{xx} = 0, \qquad 0 < x < L, \qquad 0 < t < T. \qquad (2.1)$$

$$v(x,0) = f(x), \qquad 0 < x < L \qquad (2.2)$$

$$\left.\begin{aligned} v(0,t) &= g(t), \\ v(L,t) &= h(t). \end{aligned}\right\} \; 0 < t < T \qquad \begin{aligned}(2.3)\\(2.4)\end{aligned}$$

Where $f \in C[0,L]$, $g, h \in C[0,T]$ has a unique solution on the rectangle $R: 0 \leq x \leq L,\ 0 \leq t \leq T$ for any $T > 0$.

Proof.- Uniqueness proof for problem (2) is equivalent to prove that the homogeneous problem

$$\begin{cases} u_t - k u_{xx} = 0 & (3.1) \\ u(x,0) = 0 & (3.2) \\ u(0,t) = 0 & (3.3) \\ u(L,t) = 0 & (3.4) \end{cases}$$

has only the trivial solution. Why?

We proceed as follows.

Multiply equ. (3.1) by u and $\int_0^L dx$

$$\int_0^L u_t u \, dx - K \int_0^L u_{xx} u \, dx = 0$$

$$\frac{d}{dt} \int_0^L \frac{u^2}{2} dx - K \left[\overset{0}{\cancel{u_x u \Big|_0^L}} - \int_0^L u_x^2 dx \right] = 0$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \underbrace{\int_0^L u^2(x,t) \, dx}_{E(t)} = -K \int_0^L u_x^2(x,t) \, dx$$

$$\Rightarrow \frac{1}{2} E'(t) = -K \int_0^L u_x^2(x,t) \, dx \Rightarrow E(t) \searrow$$

Also $\quad E(t) = \int_0^L u^2(x,t)\,dx \geq 0$

and $\quad E(0) = \int_0^L u^2(x,0)\,dx \overset{\text{I.C's}}{=} 0$

Therefore, $\quad E(t) \equiv 0 \Rightarrow u(x,t) \equiv 0$ on R.

Existence will be proved by constructing the solution.

Unstable IVP.

Hadamard's example

$$\begin{cases} u_{tt} + u_{xx} = 0, & t>0, \quad x \in \mathbb{R} \\ \underbrace{u(x,0) = 0, \quad u_t(x,0) = 0}_{\text{IC's.}}, & x \in \mathbb{R}. \end{cases}$$

Soln: $\quad u(x,t) \equiv 0$

If $\quad u(x,0) = 10^{-4}\sin(10^4 x)$ $\quad$ Small perturbation of IC's.

$$u(x,t) = 10^{-8}\sin(10^4 x)\underbrace{\sinh(10^4 t)}_{\substack{\downarrow\, t\to\infty \\ e^{10^4 t}}}$$

$\Rightarrow u(x,t) \xrightarrow[t\to\infty]{} \infty$ exponentially $\qquad$ Not well posed.