1.1 Diffusion Equations.

1.1.1 Conservation Laws.

They are balance laws

Thermodynamics: 1st Law

Change Internal Energy = Heat added + Work done on System.

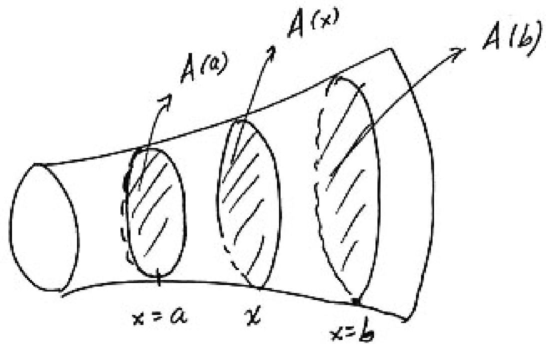
Population Biology:

Rate of change of population = Birth rate − Death rate + Migration in and out of region.

1-Dim Conservation Law

$u(x,t)$: density   amount Physical Variable / per vol   Phys. Variable = P.V. → mass, energy, Population.

Amount of Physical Variable: APV.



Assumption: 1) u is constant in any cross section of tube

2) $I = [a,b]$. arbitrary interval. of the real line $\mathbb{R}$.

Total amount of P.V. in I $= \int_a^b u(x,t)\, A(x)\, dx$

3) Motion of P.V. occurs in the axial direction

Define $\phi(x,t)$: Amount of the P.V. u flowing through cross section at x at time t, per unit of area, per unit of time.

$\overset{\|}{\text{Flux}}$

$[\phi] = $ A.P.V. / per unit of area * per unit time

4) $\phi > 0$ if flow is in the positive x-direction.

$\phi < 0$ " " " in the negative x-direction.

Net rate of flow inside I:

$$\begin{array}{l}\text{Net rate}\\ \text{of flow}\\ \text{into } I\end{array} = A(a)\,\phi(a,t) - A(b)\,\phi(b,t)$$

Sources or Sinks inside I: $f(x,t,u)$.

$$[f(x,t,u)]: \frac{\text{Amount of P.V.}}{\text{per unit Vol} * \text{per unit of time}}$$

$f>0$ is a Source

$f<0$ is a Sink.

Conservation Law:

$$\begin{array}{c}\text{Rate of change}\\ \text{of the total A.P.V}\\ \text{in } I\end{array} = \left(\underbrace{\begin{array}{c}\text{Net rate of}\\ \text{flow inside } I\end{array}}\right) + \left(\begin{array}{c}\text{Rate at which}\\ \text{A.P.V. is produced}\\ \text{or destroyed inside } I.\end{array}\right)$$

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)\,A(x)\,dx = A(a)\,\phi(a,t) - A(b)\,\phi(b,t) + \int_a^b f(x,t,u)\,A(x)\,dx} \quad (3.1)$$

Conservation Law in integral form.

Assuming that u, ϕ, A are sufficiently smooth.

i) $$\int_a^b \frac{\partial}{\partial x}\left(A(x)\,\phi(x,t)\right) dx = A(b)\,\phi(b,t) - A(a)\,\phi(a,t).$$

ii) $$\frac{d}{dt}\int_a^b A(x)\,u(x,t)\,dx = \int_a^b A(x)\frac{\partial u}{\partial t}(x,t)\,dx$$ (Assuming the cross sectional area does not change in time)

Subst. in (3.1)

$$\int_a^b \left[\frac{\partial u}{\partial t}A(x)(x,t) + \frac{\partial}{\partial x}(A\phi)(x,t) - A(x)f(x,t,u(x,t))\right]dx = 0$$

for any interval $I = [a,b]$

Then,

$$A(x)\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(A\phi) = f * A(x)$$

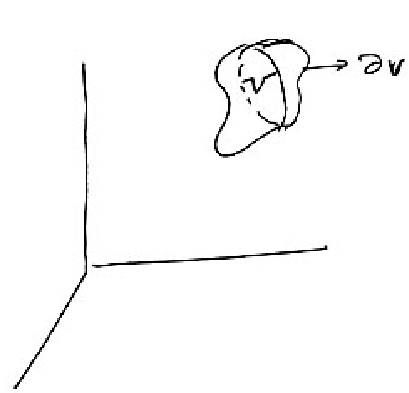
Assuming $A(x) =$ Constant.

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = f.}\,, \qquad x \in \mathbb{R},\ t > 0.$$

Conservation Law in integral form.

u, ϕ unknowns, but f is given.

Higher dimensions:



V: arbitrary region of $\mathbb{R}^3$

∂V: boundary (smooth surface) of V.

$$\text{Total Amount of P.V. in } V = \int_V u(\vec{x},t) \underbrace{dx\,dy\,dz}_{dv}$$

Flux is a vector $\vec{\phi}(\vec{x},t)$

Net outward flux of $\vec{u}$ through ∂V is given by

$$\int_{\partial V} \vec{\phi}(\vec{x},t) \cdot \hat{n}(\vec{x})\, ds.$$

Rate at which $u(\vec{x},t)$ is produced or destroyed inside V

$$\int_V f(\vec{x},t,u)\, dv$$

Conservation Law in integral form:

$$\boxed{\frac{d}{dt}\int_V u\,dv = -\int_{\partial V} \vec{\phi}.\hat{n}\,ds + \int_V f\,dv} \qquad (6.1)$$

If u and $\vec{\phi}$ suff. smooth

$$\int_V \nabla\cdot\vec{\phi}\,dv = \int_{\partial v} \vec{\phi}.\hat{n}\,ds$$

Subst. on (6.1).

$$\int_V \frac{\partial u}{\partial t}\,dv = -\int_V \nabla\cdot\vec{\phi}\,dv + \int_V f\,dv$$

Since V is arbitrary

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{\phi} = f, \qquad \vec{x}\in\mathbb{R}^3,\ t>0.$$