## 10.6.5 2-D Fourier Transform. Heat Equation

$$\begin{cases} u_t = k\,(u_{xx} + u_{yy}) & -\infty < x, y < \infty,\ t > 0 \\ u(x,y,0) = f(x,y) \\ u(x,y,t) \longrightarrow 0, & \text{when } x, y \to \pm\infty. \end{cases}$$

Sep of Vabs. lead to

$$u(x,y,t) = \phi(x,y)\, h(t)$$

$$h'(t) = -\lambda k\, h(t) \Rightarrow \boxed{h(t) = c\, e^{-\lambda k t}}$$

$$\begin{cases} \nabla^2 \phi + \lambda \phi = 0 \\ |\phi(x,y)| < \infty, & \text{when } x, y \to \pm\infty \end{cases}$$

Further Sep. : $\phi(x,y) = V(x)\, W(y)$

$$V'' W + V W'' + \lambda V W = 0$$

$\frac{1}{VW}$ $$\frac{V''}{V} + \frac{W''}{W} + \lambda = 0 \Rightarrow \frac{V''}{V(x)} + \lambda = -\frac{W''(y)}{W(y)} = +\mu$$

$$\Rightarrow \quad V'' + (\lambda - \mu) V = 0 \qquad W'' + \mu W = 0 \Rightarrow \mu > 0 \quad W_\mu(y) = e^{-i\omega_2 y} \quad \omega_2 = \pm\sqrt{\mu}$$

$$V_{\lambda,\mu}(x) = e^{-i\omega_1 x} \quad \omega_1 = \pm\sqrt{\lambda - \mu} > 0$$

$$\Rightarrow \lambda = \omega_1^2 + \omega^2$$

Therefore,

$$\boxed{\phi(x,y) = e^{-i\omega_1 x}\, e^{-i\omega_2 y}}, \quad \omega_1, \omega_2 \in \mathbb{R}$$

Therefore, the product Soln:

$$\mathcal{U}(x,y,t) = e^{-i\omega_1 x} e^{-i\omega_2 y} e^{-\kappa(\omega_1^2+\omega_2^2)t}$$

Using Superposition pple with continuous Spectrum

$$\boxed{\mathcal{U}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_1,\omega_2)\, e^{-\omega_1 x} e^{-\omega_2 y} e^{-\kappa(\omega_1^2+\omega_2^2)t}\, d\omega_1\, d\omega_2} \qquad (2.0)$$

The initial condition implies

$$f(x,y) = \mathcal{U}(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(\omega_1,\omega_2)\, e^{-\omega_1 x} e^{-\omega_2 y}\, d\omega_1\, d\omega_2$$

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It can be easily Shown that

$$\boxed{\tilde{F}(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i\omega_1 x} e^{i\omega_2 y}\, dx\, dy} \qquad (2.0.1)$$

if $f(x,y)$ has Fourier Transf. in each Variable $x$ and $y$.

**Proof:-** First Consider the F.T of $f$ w.r.t. $x$ ($y$ Fixed)

$$F(\omega_1, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,y)\, e^{i\omega_1 x}\, dx \qquad (2.1)$$

and its inverse

$$f(x,y) = \int_{-\infty}^{\infty} F(\omega_1,y)\, e^{-i\omega_1 x}\, d\omega_1 \qquad (2.2)$$

Since $F(\omega, y)$ is also a function of "y", it is possible to find its F.T.

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1, y)\, e^{i\omega_2 y}\, dy \qquad (3.1)$$

and

$$F(\omega, y) = \int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2)\, e^{-i\omega_2 y}\, d\omega_2 \qquad (3.2)$$

Combining (2.1) - (2.2) with (3.1) - (3.2).

2-D Fourier Transf. Pair

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i\omega_1 x}\, e^{i\omega_2 y}\, dx\, dy \qquad (3.3)$$

Same as (2.0.1)

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2)\, e^{-i\omega_2 y}\, e^{-i\omega_1 x}\, d\omega_2\, d\omega_1 \qquad (3.4)$$

Same as (2.0).

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Alternative notation introducing the "wave number vector"

$$e^{i\omega_1 x}\, e^{i\omega_2 y} = e^{i(\omega_1 x + \omega_2 y)} = e^{i\vec{\omega}\cdot\vec{r}}$$

where $\vec{r} = (x, y)$, $\hat{\omega} = (\omega_1, \omega_2)$

The lines $\omega_1 x + \omega_2 y = \text{Const}$ are called wave front

the normal direction to this lines is in the direction of the gradient

$$\nabla(\omega_1 x + \omega_2 y) = (\omega_1, \omega_2)$$

Thus, this vector $(\omega_1, \omega_2)$ is the direction of propagation of the wave.

The magnitude

$$\omega = |\vec{\omega}|$$

is the number of waves (full period) in $2\pi$ distance along the direction of propagation.

This is why $\vec{\omega}$ is called "wave number vector"

and $\omega = |\vec{\omega}|$ the "wave number"

Using this notation the pair (3.3)-(3.4) can be written as

$$F(\vec{\omega}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r})\, e^{i\vec{\omega}\cdot\vec{r}}\, d^2\vec{r}$$

$$f(\vec{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{\omega})\, e^{-i\vec{\omega}\cdot\vec{r}}\, d^2\omega$$

## 2-D Fourier Transform of Derivatives

1) $$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(\vec{r},t)\, e^{i\vec{\omega}\cdot\vec{r}}\, d^2 r =$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\vec{r},t)\, e^{i\vec{\omega}\cdot\vec{r}}\, d^2\vec{r}\right] = \frac{\partial}{\partial t}\left[U(\omega_1,\omega_2,t)\right] = \frac{\partial}{\partial t}\left(\mathcal{F}[u]\right)$$

2) $$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,y,t)\, e^{i(\omega_1 x + \omega_2 y)}\, dx\, dy$$

then,

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(\underbrace{\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}(x,y,t)\, e^{i\omega_1 x}\,dx}_{=-i\omega_1 U(\omega_1,y,t)}\right)dy =$$

fix

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} -i\omega_1 U(\omega_1,y,t)\,dy = -i\omega_1 U(\omega_1,\omega_2,t)$$

$$= -i\omega_1 \mathcal{F}[u].$$

Similarly,

$$\mathcal{F}\left[\frac{\partial u}{\partial y}\right] = -i\omega_2 \mathcal{F}[u].$$

$$\mathcal{F}[\nabla^2 u] = \mathcal{F}[u_{xx}] + \mathcal{F}[u_{yy}] = -\omega_1^2 U(\omega_1,\omega_2,t) - \omega_2^2 U(\ )$$

$$= -\omega^2 U(\omega_1,\omega_2,t).$$

$$= -\omega^2 \mathcal{F}[u]$$

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## Application to Heat Equ.

The equ. is transformed into

$$\frac{dU}{dt} = -k\omega^2 U, \qquad U = U(\omega_1,\omega_2,t) = \mathcal{F}[u]$$

or $$U(\vec{\omega},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\, e^{i\vec{\omega}\cdot\vec{r}}\,dx\,dy$$

Soln:

$$\boxed{U(\vec{\omega},t) = A(\vec{\omega})\, e^{-k\omega^2 t}}$$

Using I.C.

$$A(\vec{\omega}) = U(\vec{\omega},0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overset{f(x,y)}{\overset{\shortparallel}{u(x,y,0)}}\, e^{i\vec{\omega}\cdot\vec{r}}\, dx\,dy$$

$$= F(\vec{\omega})$$

$\therefore$

$$\boxed{U(\vec{\omega},t) = F(\vec{\omega})\, e^{-\kappa\omega^2 t}} \qquad (6.1)$$

## Convolution thm in 2-D.

If $\quad H(\vec{\omega}) = F(\vec{\omega})\, G(\vec{\omega})$ and

$$\mathcal{F}^{-1}[F(\vec{\omega})] = f(x,y) = f(\vec{r})$$

$$\mathcal{F}^{-1}[G(\vec{\omega})] = g(x,y) = g(\vec{r})$$

then

$$\boxed{h(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r}_0)\, g(\vec{r}-\vec{r}_0)\, d^2\vec{r}_0 \quad = \text{same interchanging } f \leftrightarrow g.} \qquad (6.2)$$

Proof:-

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\vec{\omega})\, G(\vec{\omega})\, e^{-i\vec{\omega}\cdot\vec{r}}\, d^2\omega$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{\omega}) \left( \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r}_0)\, e^{i\vec{\omega}\cdot\vec{r}_0}\, d^2\vec{r}_0 \right) e^{-i\vec{\omega}\cdot\vec{r}}\, d^2\omega$$

changing order of integr.

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r}_0) \left( \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{\omega})\, e^{-i\vec{\omega}\cdot(\vec{r}-\vec{r}_0)}\, d^2\omega}_{\downarrow} \right) d^2\vec{r}_0$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r}_0)\, g(\vec{r}-\vec{r}_0)\, d^2\vec{r}_0 .$$

Back to Soln. in freq. domain for heat equ.

In order to apply convolution thm,

We need the inverse 2-D F.T. of $e^{-\kappa\omega^2 t}$, let's call it $g(x,y)$

We already have the I.F.T. for $F(\vec{\omega}) \longrightarrow f(\vec{r}) = f(x,y)$

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\overbrace{\kappa\omega^2 t}^{-\kappa\omega_1^2 t - \kappa\omega_2^2 t}} e^{-i\overbrace{\vec{\omega}\cdot\vec{r}}^{=\omega_1 x + \omega_2 y}} d\omega_1\, d\omega_2 =$$

$$= \int_{-\infty}^{\infty} e^{-\kappa\omega_1^2 t} e^{-i\omega_1 x} d\omega_1 \int_{-\infty}^{\infty} e^{-\kappa\omega_2^2 t} e^{-i\omega_2 y} d\omega_2 =$$

$$= \sqrt{\frac{\pi}{\kappa t}}\, e^{-x^2/4\kappa t} \sqrt{\frac{\pi}{\kappa t}}\, e^{-y^2/4\kappa t} = \frac{\pi}{\kappa t}\, e^{-\frac{(x^2+y^2)}{4\kappa t}}$$

Therefore, applying convol. thm,

$$u(x,y,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_0,y_0)\frac{\pi}{\kappa t}\, e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa t}}\, dx_0\, dy_0$$

or

$$\boxed{u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_0,y_0)\left(\frac{1}{4\pi\kappa t}\, e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa t}}\right) dx_0\, dy_0}$$

influence function or fund. soln.