Divergence Theorem.

Concept of Region:– An open set containing all or some of the points forming its boundary.

Def.– A closed surface $\partial\Omega$ that consists of a finite number of smooth pieces joined together at the boundaries (curves defining boundaries) is called piecewise-smooth surface.

Def.– By a smooth surface S, we mean

$$S: \vec{r} = \vec{r}(u,v), \quad (u,v) \in D.$$

Such that $\vec{r}(u,v)$ is continuously differentiable and its unit normal vector $\hat{n}(\vec{x}_s)$ is continuous on S.

Theorem.– a) Ω is a bounded region.

b) $\partial\Omega$ is the closed piecewise smooth surface of Ω.

c) $F(\vec{x})$ is continuous on $\Omega \cup \partial\Omega$.

d) $F(\vec{x})$ is continuously differentiable in Ω.

$\Big\}\ F(\vec{x}) \in C^1(\Omega) \cap C(\bar{\Omega})$

e) $\hat{n}(\vec{x}_s)$ is the unit outer normal vector to Ω at $\vec{x}_s$.

Then,

$$\iiint_{\Omega} \nabla \cdot \vec{F}(\vec{x})\, dv = \iint_{\partial\Omega} (\vec{F} \cdot \hat{n})(\vec{x}_s)\, ds.$$

<u>Green's formula:</u> $u, v \in C^2(V) \cap C^1(\bar{V})$

$u(x,y,z)$ and $v(x,y,z)$ continuosly diff in $V+\partial V$ and having continuos 2^{nd} derivatives in V.

Then

$$\int_V u \nabla^2 v \, dv = \int_{\partial V} u \frac{\partial v}{\partial n} ds - \int_V [\nabla u \cdot \nabla v] \, dv$$

<u>Proof.</u>- $\nabla \cdot (u \nabla v) = u \nabla^2 v + \nabla u \cdot \nabla v$

Then $\int_V \nabla \cdot (u \nabla v) \, dv = \int_V u \nabla^2 v \, dv + \int_V \nabla u \cdot \nabla v \, dv$

but also $\int_V \nabla \cdot (u \nabla v) \, dv = \int_{\partial V} u \frac{\partial v}{\partial n} ds$

Therefore, $\int_V u \nabla^2 v \, dv + \int_V \nabla u \cdot \nabla v \, dv = \int_{\partial V} u \frac{\partial v}{\partial n} ds$

Green's 2^{nd} theorem or 2^{nd} identity.

V: A Volume bounded by a sufficiently smooth Surface ∂V.

$u(\vec{x}), v(\vec{x})$: Continuously differentiable in $V + \partial V$.
Continuous 2^{nd} derivatives in V.

$$\int_V [u \nabla^2 v - v \nabla^2 u] \, dv = \int_{\partial V} \left[u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] ds.$$

Discuss cylindrical and spherical coordinates.

Laplace

In cylindrical coordinates:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}$$

In spherical coordinates:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2 u}{\partial \theta^2}$$

Consider the BVP for Poisson's equ.

Find $u \in C^2(V) \cap C^1(\bar{V})$ Satisfying

$$\begin{cases} \Delta u = f(\vec{x}), & \vec{x} \in V \quad (6.1) \\ u(\vec{x}) = h(\vec{x}), & \vec{x} \in \partial V \quad (6.2) \end{cases}$$

where V is an open set bounded by the Smooth Surface ∂V, f is continuous on $V + \partial V$ and h is continuous on ∂V.

<u>Theorem:</u> If the BVP (6.1) and (6.2) described above has a this Solution is unique.

<u>Proof:-</u> To prove uniqueness, consider two solutions $u_1(x)$ and $u_2(x)$ of problem (6.1)-(6.2). Therefore,

$$\omega(\vec{x}) \equiv u_1(\vec{x}) - u_2(\vec{x})$$

Satisfies the homogeneous problem

$$\begin{cases} \Delta \omega = 0, & \vec{x} \in V \\ \omega(\vec{x}) = 0, & \vec{x} \in \partial V \end{cases}$$

Now, if u and v in Green's first identity are replaced by w, then

$$\int_V w \cancel{\nabla^2 w}^{\,0}\, dv + \int_V \nabla w \cdot \nabla w \, dv = \int_{\partial V} \cancel{w}^{\,0} \frac{\partial w}{\partial n}\, ds$$

then,

$$\int_V |\nabla w|^2 \, dv = 0 \Rightarrow \nabla w(\vec{x}) \equiv 0, \quad \text{for all } \vec{x} \in V$$

this leads to

$$w(\vec{x}) \equiv C \text{ (const.) in } V$$

But w is conts on $\bar{V}$, then

$$w(\vec{x}) \equiv C \text{ in } \bar{V}$$

Also, $w(\vec{x}) = 0, \quad \vec{x} \in \partial V$

Then, using cont. $w(\vec{x}) \equiv 0, \quad x \in \bar{V},$

Which also implies that

$$u_1(\vec{x}) = u_2(\vec{x}) \quad \text{in } \bar{V}$$

Consider the Neumann problem

$$\text{BVP}\begin{cases} \Delta u = f(\vec{x}), & \vec{x} \in V \quad (8.1) \\ \dfrac{\partial u}{\partial n} = g(\vec{x}), & \vec{x} \in \partial V \quad (8.2) \end{cases}$$

Where V, ∂V, and u have the same properties described in the previous theorem. And $g(\vec{x})$ is Continuous in ∂V.

<u>Theorem.-</u> (Compatibility Condition).

Under the above conditions on u, g, V and ∂V the following relationship between f and g holds

$$\int_V f(\vec{x})\, dv = \int_{\partial V} g(\vec{x})\, ds$$

<u>Proof.-</u> (different than book).

From (8.1) $\quad \nabla\cdot(\nabla u) = f \Rightarrow \int_V \nabla\cdot\nabla u\, dv = \int_V f(\vec{x})\, dv$

Applying divergence theorem $\int_V \nabla\cdot\nabla u\, dv = \int_{\partial V} \frac{\partial u}{\partial n}\, ds$

Thus, $\int_{\partial V} \frac{\partial u}{\partial n}\, ds = \int_V f(\vec{x})\, dv \overset{\text{Using B.C.}}{\Rightarrow} \int_{\partial V} g(\vec{x})\, ds = \int_V f(\vec{x})\, dv.$