Fluid Flow Past a Circular Cylinder

Viscous, Incompressible and Steady.

$$\begin{cases} \rho\left[\cancel{\frac{\partial \vec{v}}{\partial t}} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla P + \mu \nabla^2 \vec{v} \\ \nabla\cdot\vec{v} = 0 \end{cases}$$

B.C. at the Cylinder: No penetration

$$\vec{v}\cdot\hat{n} = 0, \qquad X^2 + Y^2 = a^2.$$

B.C. at infinity: Uniform flow

$$\vec{v}(x,y) \xrightarrow[R\to\infty]{} (v_\infty, 0)$$

BVP

Scaling

Nondimensional analysis

$$\vec{v} = v_\infty \vec{u}$$

$$P = \rho v_\infty^2 p$$

$$\vec{X} = a\vec{x}$$

$$(0.1)\quad (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u}$$

$$(0.2)\quad \nabla\cdot\vec{u} = 0$$

$$(0.3)\quad \vec{u}\cdot\hat{n} = 0 \quad \text{on} \quad x^2+y^2=1$$

$$(0.4)\quad \vec{u}(x,y) \xrightarrow[r\to\infty]{} (1,0).$$

Nondimensional BVP.

2) c) Nondimensional, steady, incompressible, Viscous Navier-stokes equation at large Reynolds number:

$$\begin{cases} (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u} & (10.1) \\ \nabla\cdot\vec{u} = 0 & (10.2) \end{cases}$$

Applying Curl on both sides of (10.1)

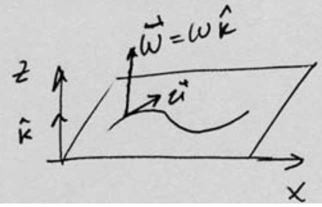
$$\nabla\times\left((\vec{u}\cdot\nabla)\vec{u}\right) = -\cancelto{0}{\nabla\times(\nabla p)} + \frac{1}{Re}\nabla\times(\nabla^2\vec{u})$$

Using 2) ⓐ and ⓑ.

$$\boxed{(\vec{u}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{u} = \frac{1}{Re}\nabla^2(\nabla\times\vec{u}) = \frac{1}{Re}\nabla^2\vec{\omega}.} \quad (10.5).$$

d) For 2-dimensional flow

$$\vec{\omega} = \nabla\times\vec{u} = \omega\hat{k}$$



Then (10.5) reduces to

$$(\vec{u}\cdot\nabla)\vec{\omega} = \frac{1}{Re}\nabla^2\vec{\omega}$$

Subst. $\vec{\omega}(x,y) = \omega(x,y)\hat{k}$, we arrive to the scalar equation

$$\boxed{\vec{u}\cdot\nabla\omega = \frac{1}{Re}\nabla^2\omega} \qquad (10.6)$$

Assuming $Re \gg 1$ (large Reynolds number),
we expect to find an approximate solution to (10.6) in the form of an asymptotic expansion of $\vec{u}$ in terms of the small parameter $\varepsilon = \frac{1}{Re}$,

for example,

$$\vec{u} = \vec{u}^0 + \varepsilon\vec{u}^1 + \varepsilon^2\vec{u}^2 + \cdots$$

$$\Rightarrow \quad \vec{\omega} = \nabla\times\vec{u} = \nabla\times\vec{u}_0 + \varepsilon\,\nabla\times\vec{u}^1 + \varepsilon^2\,\nabla\times\vec{u}^2 + \cdots$$

$$= \vec{\omega}^0 + \varepsilon\vec{\omega}^1 + \varepsilon^2\vec{\omega}^2 + \cdots$$

Substitution into (10.6) leads to

$$\vec{u}^\circ\cdot\nabla\omega^\circ + \varepsilon\,\vec{u}^\circ\cdot\nabla\omega' + \varepsilon\,\vec{u}'\cdot\nabla\omega^\circ + \cdots = \varepsilon\nabla^2\omega^\circ + \varepsilon^2\nabla^2\omega' + \cdots$$

$$\Rightarrow \vec{u}^\circ\cdot\nabla\omega^\circ + \mathcal{O}(\varepsilon) = \varepsilon\nabla^2\omega^\circ + \mathcal{O}(\varepsilon^2)$$

Therefore, $\boxed{\vec{u}^\circ\cdot\nabla\omega^\circ = 0}$ Equation satisfied by the leading order term. (10.7).

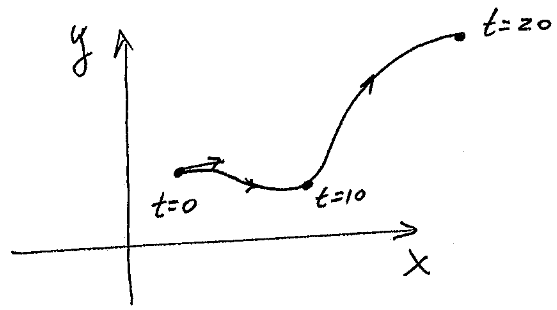
Next term of order ε:

$\boxed{\vec{u}^\circ\cdot\nabla\omega' + \vec{u}'\cdot\nabla\omega^\circ = \nabla^2\omega^\circ}$ linear equation.

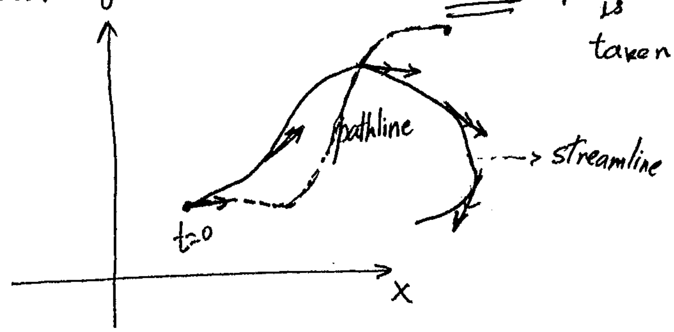
Rmk: Not enough to consider a regular exp. of soln. for a uniformly valid soln. everywhere.

STREAMLINES, PATHLINES, and STREAMFUNCTIONS.

Definition.- A pathline is the trajectory of a single particle of fluid.



Definition.- A streamline is a curve everywhere parallel to the direction of the flow.



Graphs in the plane or space for the pathlines and streamlines are generally different. However, if flow is steady they are identical.

Stream function for Incompressible two-dimensional flow

First, let's consider a differentiable function of two variables,

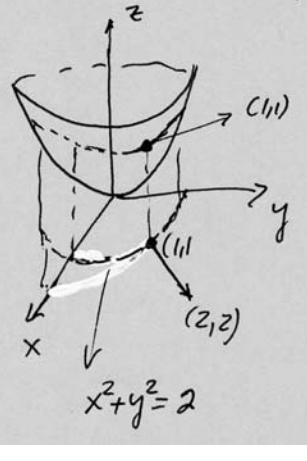
$$z = f(x,y), \qquad (x,y) \in \mathcal{D}.$$

For any $(x_0, y_0) \in \mathcal{D}$

$$f(x,y) = f(x_0,y_0) = z_0$$

defines a curve called level curve.

Example: $z = f(x,y) = x^2 + y^2$



$$f(x,y) = x^2 + y^2 = \underset{= f(1,1)}{1^2 + 1^2} = 2$$

Now, $\nabla f(1,1) = (2x, 2y)\big|_{(x,y)=(1,1)} = (2,2)$

$\nabla f(1,1) \perp$ tangent vector at (1,1) of level curve

Streamfunctions for two-dimensional fluids:

Consider a function $\psi(x,y)$ such that if $u(x,y) = (u_1(x,y), u_2(x,y))$, then

$$u_1(x,y) = \frac{\partial \psi}{\partial y}(x,y) \quad \text{and} \quad u_2(x,y) = -\frac{\partial \psi}{\partial x}(x,y) \qquad (6.1)$$

or equivalently, we are assuming that

$$\vec{u}(x,y) = \nabla_\times (\psi \hat{k})(x,y) \qquad (6.2)$$

Since

$$\nabla_\times (\psi \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j} \overset{(6.1)}{=} u_1\hat{i} + u_2\hat{j}.$$

this function ψ has the following properties:

a) $\vec{u}\cdot\nabla\psi = (u_1,u_2)\cdot\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) =$

$$= \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)\cdot\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0$$

$\Rightarrow \quad \vec{u} \perp \nabla\psi$, for all (x,y).

But $\nabla\psi(x,y)$ is also orthogonal to the level curves of $\psi(x,y)$. Then, the level curves,

$$\psi(x,y) = \psi_0 \tag{7.1}$$

are everywhere tangent to the vector field $\vec{u}$.

Therefore, the level curves (7.1) are "streamlines" of the fluid motion and the function $\psi = \psi(x,y)$ is called Stream-function

b) Also, it is easy to prove, that $\vec{u}$ represented by (6.2) in terms of ψ, satisfies the continuity equ. $\nabla . \vec{u} = 0$

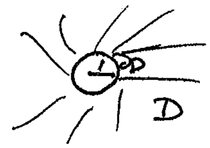
In fact,

$$\nabla . \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) =$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \quad ✓ \tag{7.2}$$

The reason for introducing ψ in this problem is to arrive (if possible) to an easier BVP for ψ that once solved provides a soln. for the BVP for u defined by (10.7), (0.2) - (0.4). As shown above (0.2) is already satisfied.

Next, we will show that the flow described by BVP:



$$\begin{cases} \vec{u} \cdot \nabla \omega = 0, & \text{in } D \quad (8.1) \\ \nabla \cdot \vec{u} = 0, & \text{in } D \quad (8.2) \\ \vec{u} \cdot \hat{n} = 0, & \text{on } \partial D \quad (8.3) \\ \vec{u}(x,y) \xrightarrow[r \to \infty]{} (1,0) & \quad (8.4) \end{cases}$$

(I) First, it's irrotational: $\omega(x,y) \equiv 0$ for all $(x,y) \in D$

(II) Secondly, by solving the BVP below for ψ, a soln. $\vec{u}(x,y)$ of (8.1)-(8.4) can be obtained.

$$\begin{cases} \nabla^2 \psi = 0, & \text{in } D \\ \psi(1,\theta) = 0, & \text{on } \partial D \\ \psi(r,\theta) \xrightarrow[r \to \infty]{} r \sin\theta \end{cases}$$

(I) Vector field $\vec{u}$ is irrotational

Since $\vec{u} \cdot \nabla\omega = 0$, for all (x,y)

$\nabla\omega \perp \vec{u}$ " " "

Therefore, level curves $\omega(x,y) = \omega_0$ are tangent to $\vec{u}$ and parallel to the streamlines $\psi(x,y) = \psi_0$ everywhere

Now, for any (x_0, y_0) in D, there are level curves

$$\ell_1 : \omega(x,y) = \omega(x_0,y_0) = \omega_0$$

and $\ell_2 : \psi(x,y) = \psi(x_0,y_0) = \psi_0$

that pass through (x_0,y_0). According to lemma _ in Appendix the level curves ℓ_1 and ℓ_2 coincide.

Therefore, there is a one-to-one correspondence btw.

$$\omega = \omega_0 \longrightarrow \psi = \psi_0$$

level curves.

As a consequence, $\boxed{\omega(x,y) \text{ is constant along streamlines.}}$

Key result.

On the other hand,

$W(x,y) = 0$, for all (x,y) on streamlines that begin upstream in the uniform flow region.

$$\left(\begin{aligned} &\text{Since} \quad \vec{u}(x,y) \xrightarrow[r\to\infty]{} (1,0) \\ &\Rightarrow \quad \vec{\omega}(x,y) = \omega(x,y)\,\hat{k} = \nabla\times\vec{u}(x,y) \xrightarrow[r\to\infty]{} 0\,\hat{k} \end{aligned} \right)$$

Therefore, $\omega(x,y) \equiv 0$, along all streamlines

Now, for any point $(x,y) \in D$, there exists a streamline that passes through it, then $\omega(x,y) \equiv 0$ everywhere in D.

Moreover, from B.C. at infinity

$$\vec{u}(x,y) \to (1,0)$$

we obtain $\vec{\omega}(x,y) = \omega(x,y)\,\hat{k} = \nabla \times \vec{u}(x,y) \underset{r\to\infty}{\longrightarrow} 0\,\hat{k}$

Since $\omega(x,y)$ is constant along streamlines

$$\omega(\psi) = 0 \quad \text{or} \quad \boxed{\omega(x,y) = 0,}$$

for all (x,y). on streamlines that begin upstream in the uniform flow region. (10.9)

The flow is irrotational

Since $\omega(x,y) = \nabla \times \vec{u}(x,y)$ (11.1)

and $\vec{u}(x,y) = \nabla \times (\psi(x,y)\,\hat{k})$ (11.2)

Combining these two equations.

Boundary Conditions:

The B.C. $\left.\vec{u}\cdot\hat{n}\right|_{r=1}=0$ → Condition of no penetration on the circular boundary of the cylinder $r=1$.

reduces to

$$\boxed{u_r(1,\theta)=\vec{u}\cdot\hat{r}(1,\theta)=0}, \quad \text{since } \hat{n}=\hat{r}. \qquad (12.1)$$

u_r ↓ radial component of the velocity vector.

Now,

$$u(x,y)=\nabla_x\left(\psi(x,y)\hat{k}\right)$$

CARTESIAN COORDINATES:

$$u_1(x,y)=\frac{\partial\psi}{\partial y}(x,y), \qquad u_2(x,y)=-\frac{\partial\psi}{\partial x}(x,y)$$

Polar Coordinates:

$$u_r(r,\theta)=\frac{1}{r}\frac{\partial\psi}{\partial\theta}(r,\theta), \qquad u_\theta(r,\theta)=-\frac{\partial\psi}{\partial r}(r,\theta)$$

Therefore, equ. (12.1) in terms of the streamfunction is given by

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta}(1,\theta)=0 \xRightarrow{\text{Integrating}} \boxed{\psi(1,\theta)=\text{Const.}}$$

In particular, if

$$\boxed{\psi(1,\theta)=0} \qquad (12.2)$$

the B.C. (12.1) is satisfied.

The other B.C. when $r \to \infty$ (0.4)

$$\vec{u}(x,y) \xrightarrow[r\to\infty]{} (1,0)$$

is equivalent to

$$\left(\frac{\partial\psi}{\partial y}(x,y), -\frac{\partial\psi}{\partial x}(x,y)\right) \xrightarrow[r\to\infty]{} (1,0).$$

So when $r \to \infty$

$$\frac{\partial\psi}{\partial y}(x,y) \simeq 1, \qquad \frac{\partial\psi}{\partial x}(x,y) \simeq 0$$

Integrating $\psi(x,y) = y + C(x)$

$$\Rightarrow 0 \simeq \frac{\partial\psi}{\partial x} = C'(x) \Rightarrow \quad C(x) \equiv \text{Const.}$$

and $$\boxed{\psi(x,y) \xrightarrow[r\to\infty]{} y + \text{Const.}}$$

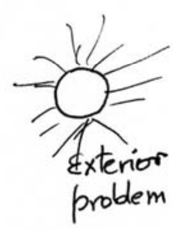
In particular, if

$$\boxed{\psi(x,y) \xrightarrow[r\to\infty]{} y} \Rightarrow \quad \vec{u}(x,y) \xrightarrow[r\to\infty]{} (1,0)$$

(13.1)

Thus, our new BVP for the stream function $\psi = \psi(x,y)$ is given by

$$\begin{cases} \nabla^2 \psi = 0, \quad \text{on } \Omega \equiv \{(x,y) : x^2+y^2>1\} & (14.1) \\ \psi(1,\theta) = 0 & (14.2) \\ \psi(r,\theta) \underset{r\to\infty}{\longrightarrow} y & (14.3) \end{cases}$$



This BVP is not equivalent to the original BVP

Why? – Highest order derivative term has been dropped (for example)

However, it retains some of the physical properties of the original BVP.

Solution for BVP (14) can be easily obtained by separation of variables. In fact,

$$\boxed{\psi(r,\theta) = C_1 \ln r + \left(r - \frac{1}{r}\right)\sin\theta}$$

are solutions of (14) for any value of C_1.

Now from Solution (14.4), we can obtain the two components of the velocity

$$u_r(r,\theta) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta(r,\theta) = -\frac{\partial\psi}{\partial r}$$

Therefore,

$$\boxed{u_r(r,\theta) = \frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta} \qquad (15.1)$$

and

$$\boxed{u_\theta(r,\theta) = -\frac{C_1}{r} - \left(1 + \frac{1}{r^2}\right)\sin\theta} \qquad (15.2)$$

We can also obtain the pressure. Going back to Navier–Stokes equation, under steady, inviscid, irrotational conditions and incompressibility.

$$\rho\left[\cancel{\frac{\partial\vec{u}}{\partial t}} + (\vec{u}\cdot\nabla\vec{u})\right] = -\nabla p + \cancel{\mu\nabla^2\vec{u}} \qquad Re \gg 1 \text{ if } \mu \ll 1.$$

$$\therefore \quad \boxed{\rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla p} \qquad (15.3)$$

Now, $(\vec{u}\cdot\nabla)\vec{u} = \nabla\left(\frac{|\vec{u}|^2}{2}\right) - \vec{u}\times(\nabla\times\vec{u})$

$\therefore$ Equ. (15.3) reduces to

$$\rho\left[\nabla\left(\frac{|\vec{u}|^2}{2}\right) - \cancel{\vec{u}\times(\nabla\times\vec{u})}^{\,0 \text{ irrotational}}\right] = -\nabla p,$$

everywhere on the domain $x^2+y^2 > 1$.

or

$$\rho \, \nabla\left(\frac{|\vec{u}|^2}{2}\right) = -\nabla p$$

if $\rho \equiv$ Const.
$$\nabla\left[\rho \frac{|\vec{u}|^2}{2} + p\right] = 0$$

integrating, we obtain Bernoulli's equation

$$\boxed{\rho \frac{|\vec{u}|^2}{2} + p = \text{Const.}} \qquad (16.1)$$

the "Const" can be obtained from the condition at ∞.

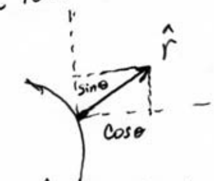
$\vec{v}_\infty = (1,0)$.

$$\rho \frac{|\vec{u}|^2}{2} + p = \rho \frac{\overbrace{|\vec{v}_\infty|^2}^{=1}}{2} + p_\infty .$$

then
$$\boxed{p = -\rho \frac{|\vec{u}|^2}{2} + \rho/2 + p_\infty .} \qquad (16.2)$$

<u>Force on the cylinder</u> $(r=1)$

opposite direction to the outer normal



$\hat{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$.

$$\vec{F}_{cyl} = \left[\int_0^{2\pi} \left(p(r,\theta)\, r\right)\left(-\hat{r}(\theta)\right) d\theta\right]_{r=1}$$

$$= -\int_0^{2\pi} p(1,\theta)\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right) d\theta \qquad (16.3)$$

Now, from (16.2).

$$p(1,\theta) = -\frac{1}{2}\rho\left(U_r^2(1,\theta) + U_\theta^2(1,\theta)\right) + \text{Const.}$$

And $U_r(1,\theta) = 0$ (boundary condition)

Thus,

$$p(1,\theta) = -\frac{1}{2}\rho\, U_\theta^2(1,\theta) + \text{Const.}$$

$$= -\frac{1}{2}\rho\left[-\frac{C_1}{r} - \left(1+\frac{1}{r^2}\right)\sin\theta\right]_{r=1}$$

Substitution into (16.3)

$$\vec{F}_{cyl} = -\int_0^{2\pi}\left\{\frac{-1}{2}\rho\left[-C_1 - (1+1)\sin\theta\right]^2 + \text{Const}\right\}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\,d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}\rho\left[C_1 + 2\sin\theta\right]^2(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\,d\theta + \int_0^{2\pi}\text{const}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\,d\theta$$

Using that $\int_0^{2\pi}\cos\theta\,d\theta = 0$, $\int_0^{2\pi}\sin\theta\,d\theta = 0$, $\int_0^{2\pi}\sin\theta\cos\theta = 0$

$\int_0^{2\pi}\sin^2\theta\cos\theta = 0$

$$\vec{F}_{cyl} = \frac{1}{2}\rho\int_0^{2\pi}4C_1\sin^2\theta\,d\theta\,\hat{\jmath} = 2\rho C_1\int_0^{2\pi}\sin^2\theta\,d\theta\,\hat{\jmath} = 2\rho C_1\pi\,\hat{\jmath}$$

$$\therefore\ \boxed{\vec{F}_{cyl} = 2\rho C_1\pi\,\hat{\jmath}} \qquad (17.1)$$

Interpretation of the results.

From Bernoulli's equation (16.2), we observe that if the magnitude $|\vec{u}|$ increases then the pressure p decreases.

So pressure is lower where the velocity is higher.

Also, from the expression for the force (17.1)

$$\boxed{F_{cyl} = 2\rho C_1 \pi \, \hat{j}.}$$

We find out that there is not force in the x-direction (Known as drag force). The only force component is in the y-direction (Known as lift).

And the lift is greater as C_1 increases.

Show Lin-Segel comments on this (pp. 557)

This result is known as D'Alembert paradox.

Drawback in potential flow model is

1) Ignoring viscosity
2) Incomplete B.C. at the cylinder. No-slip condition, $\vec{u}=0$ on $r=1$, needs to be added.