Incompressible and Inviscid Fluid Equations

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September 13, 2010

1 The Continuity Equation

In this section we will develop the continuity equation. This is the basic equation that will enable us to derive many other useful formulas. For this derivation we will be working with some arbitrary volume of water, which we will call D. This could be a lake, the ocean, or some more controlled system such as a tank. Later when we begin our work with waves, we will specify exactly what we want D to be.

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a velocity field, where $u_i = u_i(x, y, z, t)$ for $i = 1, 2, 3$.

The density of the water at a point (x, y, z) and at time t will be denoted as $\rho(x, y, z, t)$.

Let V be some control volume within D, and S the surface of V.

The mass of V can be written as $m = \int_V \rho(x, y, z, t)\, dV$.

The rate of change in mass for the volume V can then be written as $d/dt\left(\int_V \rho\, dV\right)$.

The rate of change of mass is also equal to then net mass that passes through the surface S, that is -$\int_S \rho\mathbf{u} \cdot \mathbf{n}\, dS$ where $\mathbf{n}$ is the outward unit normal. Hence we can write

$$\int_V \frac{\partial \rho}{\partial t} = -\int_S \rho\mathbf{u} \cdot \mathbf{n}\, dS. \tag{1.1}$$

A quick check shows that the units work out correctly $\frac{kg}{s} = \frac{(kg)(m^3)}{(m^3)(s)} = \frac{(kg)(m)(m^2)}{(m^3)(s)}$.

The integral on the left is obtained by applying Leibniz's rule. It means

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV$$

The conditions are that
$\rho(\mathbf{x}, t)$ is an integrable function on V for each $t \in \mathbb{R}^+$ and
$\frac{\partial \rho}{\partial t}(\mathbf{x}, t)$ exists and is continuous in $V \times \mathbb{R}^+$.

Remark 1.1. The negative in the equation above can be explained as follows: Adding mass to a system results in a positive change in volume, hence inflow is a positive change in volume. However, the unit normal to S points outward and must be negated to reflect a positive inflow.

The Divergence Theorem applied on (1.1) results in

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_S (\rho\mathbf{u}) \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot (\rho\mathbf{u})\, dV$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\right] dV = 0. \tag{1.2}$$

Since the volume V is arbitrary, then the integrand should be zero, *i. e.,*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \tag{1.3}$$

which is known as the continuity equation. If the fluid is incompressible, then the density is constant. Thus $\partial\rho/\partial t = 0$, and $\nabla\rho = \mathbf{0}$. Expanding (1.3) and making the suggested substitutions yields

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0\\ \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\mathbf{u} + \rho(\nabla\cdot\mathbf{u}) &= 0\\ \nabla\cdot\mathbf{u} &= 0. \qquad (1.4)\end{aligned}$$

The equation above is the continuity equation for a fluid of constant density. Assuming constant density for water is a valid assumption, which we will make for the remainder of this paper.

2 Euler's Equation of Motion

We will now use the continuity equation (1.3) to derive Euler's equation of motion which will allow us to find Bernoulli's equation. In order to derive Euler's equation of motion, we will use techniques similar to those used to derive the continuity equation. We are still considering the same domain D, velocity field $\mathbf{u}$, and volume V with a surface S.

We will also need to use the following two results.

Lemma 2.1. *If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is a continuously differentiable vector function, then*

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \langle \mathbf{u}\cdot\nabla u_1, \mathbf{u}\cdot\nabla u_2, \mathbf{u}\cdot\nabla u_3\rangle$$

Lemma 2.2. *If f is a continuously differentiable scalar function, V a bounded region in the space, and S the surface enclosing the region V, then*

$$\int_S f\mathbf{n}\,dS = \int_V \nabla f\,dV$$

where $\mathbf{n}$ is the outward unit normal.

Proof. Let f be a scalar function, and define a vector function $\mathbf{F}$ so that $\mathbf{F} = f\mathbf{c}$ for some non-zero constant vector $\mathbf{c}$. Then by Gauss's Theorem (Divergence Theorem),

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_V \operatorname{div}(\mathbf{F})\,dV\\ \int_S (f\mathbf{c})\cdot\mathbf{n}\,dS &= \int_V \operatorname{div}(f\mathbf{c})\,dV\\ \int_S f\mathbf{n}\,dS\cdot\mathbf{c} &= \int_V (\nabla f\cdot\mathbf{c}) + f\operatorname{div}(\mathbf{c})\,dV\\ \int_S f\mathbf{n}\,dS\cdot\mathbf{c} &= \int_V (\nabla f\cdot\mathbf{c})\,dV\\ \int_S f\mathbf{n}\,dS\cdot\mathbf{c} &= \int_V \nabla f\,dV\cdot\mathbf{c}\\ \left[\int_S f\mathbf{n}\,dS - \int_V \nabla f\,dV\right]\cdot\mathbf{c} &= 0. \quad \text{Since } \mathbf{c}\neq 0\text{, then}\\ \int_S f\mathbf{n}\,dS &= \int_V \nabla f\,dV\end{aligned}$$

□

Now, let's consider the law of conservation of momentum.
The momentum, I, can be written as $I = \int_V (\rho\mathbf{u})\,dV$,
and the rate of change of momentum as $dI/dt = d/dt \int_V (\rho\mathbf{u})\,dV$.

Also the net inflow of momentum is given by $-\int_S (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS$.
By the Law of Conservation of Momentum, the rate of change of momentum within the volume V equals the net inflow of momentum through the surface S plus the force $\mathbf{F}_p$ due to the pressure forces acting on the surface S and the force $\mathbf{F}_e$ due to the body forces per unit of mass acting inside the volume V. Thus we have

$$\frac{dI}{dt} = \frac{d}{dt}\int_V (\rho\mathbf{u})\,dV = -\int_S (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS + \mathbf{F}, \tag{2.1}$$

where $\mathbf{F} = \mathbf{F}_p + \mathbf{F}_e$ A quick check insures that the units work out correctly, $\frac{(kg)(m)}{(s^2)} = \frac{(kg)(m)(m^3)}{(s)(m^3)(s)} = \frac{(kg)(m)(m)(m^2)}{(m^3)(s)(s)}$.

Upon expanding (2.1) component wise, we get

$$\int_V \frac{\partial}{\partial t}(\rho\mathbf{u})\,dV = -\int_S \langle(\rho u_1\mathbf{u})\cdot\mathbf{n}, (\rho u_2\mathbf{u})\cdot\mathbf{n}, (\rho u_3\mathbf{u})\cdot\mathbf{n}\rangle\; dV + \mathbf{F}.$$

Gauss's Theorem (Divergence Theorem) applied to each component on the right hand side of the above equation yields

$$\int_V \left(\rho\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\rho}{\partial t}\mathbf{u}\right)\,dV = -\int_V \langle\nabla\cdot(\rho u_1\mathbf{u}), \nabla\cdot(\rho u_2\mathbf{u}), \nabla\cdot(\rho u_3\mathbf{u})\rangle\,dV + \mathbf{F}.$$

Using the incompressibility condition and vector calculus gives

$$\int_V \rho\frac{\partial\mathbf{u}}{\partial t}\,dV = -\int_V \langle\rho(\mathbf{u}\cdot\nabla u_1) + u_1\left(\nabla\cdot(\rho\mathbf{u})\right), \rho(\mathbf{u}\cdot\nabla u_2) + u_2\left(\nabla\cdot(\rho\mathbf{u})\right), \rho(\mathbf{u}\cdot\nabla u_3) + u_3\left(\nabla\cdot(\rho\mathbf{u})\right)\rangle\; dV + \mathbf{F},$$

Applying Lemma 2.1 leads to

$$\int_V \rho\frac{\partial\mathbf{u}}{\partial t}\,dV = -\int_V \left(\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\cdot(\rho\mathbf{u})\mathbf{u}\right)\,dV + \mathbf{F}.$$

Now, using the continuity equation (1.4) and the incompressibility condition leads to $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} = 0$, then

$$\int_V \rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]\,dV = \mathbf{F}. \tag{2.2}$$

Notice that $\mathbf{F}$ is the total force acting on the fluid, so we can break it up so that $\mathbf{F} = \mathbf{F}_p + \mathbf{F}_e$, where $\mathbf{F}_p$ is the force due to pressure, and $\mathbf{F}_e$ is some other external force. Pressure is force per unit area, so we can write

$$\mathbf{F}_p = -\int_S P\mathbf{n}\,dS \tag{2.3}$$

where $P = P(x,y,z,t)$ is the pressure. Again note the negative sign since the unit normal is outward. Also, if $\mathbf{A}_e$ represents the body forces per unit of mass, then the external force $\mathbf{F}_e$ is given by

$$\mathbf{F}_e = \int_V \rho\mathbf{A}_e\,dV \tag{2.4}$$

Substitution of (2.3), and (2.4) into (2.2) yields

$$\int_V \rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]\,dV = \int_V \rho\mathbf{A}_e\,dV - \int_S P\mathbf{n}\,dS.$$

By Lemma 2.2 (a consequence of the Divergence Theorem),

$$\int_V \rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]\,dV = \int_V \rho\mathbf{A}_e\,dV - \int_V \nabla P\,dV$$
$$\int_V \left[\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{A}_e + \nabla P\right]\,dV = 0.$$

Once again, the volume is arbitrary, so it must be the case that

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \rho \mathbf{A}_e + \nabla P = 0.$$

Or by rearrange terms, we get Euler's equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{A}_e - \frac{\nabla P}{\rho}. \tag{2.5}$$