10.6.1

Fourier Transform. 1-dim. wave equation.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, \quad -\infty < x < \infty. \\ \left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array}\right\} \text{IC's.} \end{cases}$$

$$\bar{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx$$

$$u(x,t) = \int_{-\infty}^{\infty} \bar{U}(\omega,t)\, e^{-i\omega x}\, d\omega$$

$$\mathcal{F}\,[\text{wave equ.}] \Leftrightarrow \boxed{\frac{\partial^2 \bar{U}}{\partial t^2}(\omega,t) = -c^2\omega^2\,\bar{U}(\omega,t)} \quad (*)$$

2nd order ODE in time variable

From IC's.

$$\bar{U}(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overbrace{u(x,0)}^{=f(x)}\, e^{i\omega x}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

$$\frac{\partial \bar{U}}{\partial t}(\omega,0) = 0$$

General Soln of $(*)$.

$$\bar{U}(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t).$$

$$\frac{\partial \bar{U}}{\partial t}(\omega,t) = -c\omega\, A(\omega)\, \sin(c\omega t) + c\omega B(\omega) \cos(c\omega t)$$

Since $\quad \dfrac{\partial \bar{U}}{\partial t}(\omega,0) = 0$

$$\Rightarrow B(\omega) = 0$$

$$\Rightarrow \boxed{\bar{U}(\omega,t) = A(\omega)\, \cos(c\omega t)} \quad (**)$$

Using the other IC. $\rightarrow \bar{U}(\omega,0) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$

But also $\bar{U}(\omega,0) = A(\omega) \cos(0) = A(\omega) \Rightarrow \boxed{A(\omega) = \bar{U}(\omega,0)}$ $(***)$

Product of two F.T.

$$\Rightarrow \boxed{\bar{U}(\omega,t) = \bar{U}(\omega,0) \cos(c\omega t)}$$

Applying the conv. thm. would require to know the inverse F.T. of $\cos(c\omega t)$.

Now, there is an easier way. Combining $(**)$ and $(***)$

$$\bar{U}(\omega,t) = \bar{U}(\omega,0) \cos(c\omega t)$$

$\Downarrow$ Inv. F.T.

$$u(x,t) = \int_{-\infty}^{\infty} \bar{U}(\omega,0) \cos(c\omega t)\, e^{-i\omega x}\, d\omega$$

or

$$u(x,t) = \int_{-\infty}^{\infty} \bar{v}(\omega,0)\, \frac{e^{i\omega ct} + e^{-i\omega ct}}{2}\, e^{-i\omega x}\, d\omega =$$

$\therefore$

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \bar{v}(\omega,0)\left[e^{-i\omega(x-ct)} + e^{-i\omega(x+ct)} \right] d\omega \quad (\circ)$$

On the other hand,

$$\bar{v}(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

$$\Downarrow$$

$$f(x) = \int_{-\infty}^{\infty} \bar{v}(\omega,0)\, e^{-i\omega x}\, d\omega \quad (\circ\circ)$$

$$\Rightarrow\ f(x-A) = \int_{-\infty}^{\infty} \bar{v}(\omega,0)\, e^{-i\omega(x-A)}\, d\omega$$

Therefore,

$$u(x,t) = \frac{1}{2}\left[\int_{-\infty}^{\infty} \bar{v}(\omega,0)\, e^{-i\omega(x-ct)}\, d\omega + \int_{-\infty}^{\infty} \bar{v}(\omega,0)\, e^{-i\omega(x+ct)}\, d\omega \right]$$

$\therefore$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]}$$

Two traveling waves.

Heat Equ. in a Semi-infinite interval.

$$\begin{cases} u_t = k u_{xx}, & x>0, \quad t>0 \\ u(0,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Sep of Variables:

$$\frac{dh}{dt} = -\lambda k h \Rightarrow h(t) = c e^{-\lambda k t}$$

Eigenvalue Probl.

$$\begin{cases} \phi'' + \lambda \phi = 0 \\ \phi(0) = 0, \quad |\phi(x)| < \infty \end{cases}$$

Non trivial Solns. only for $\lambda > 0$

$$\phi(x) = c_1 \sin\sqrt{\lambda}\, x = c_1 \sin \omega x.$$

Product Solns:

$$\boxed{u(x,t) = A \sin \omega x \, e^{-kt\omega^2}}, \quad x>0, \ t>0$$

Generalized Pple of Superp.

$$\boxed{u(x,t) = \int_0^\infty A(\omega) \sin(\omega x)\, e^{-k\omega^2 t}\, d\omega}$$

I.C. is satisfied if

$$f(x) = \int_0^\infty A(\omega)\sin\omega x \, d\omega \qquad \text{"Fourier sine transform" of } f(x).$$

We will show next that

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx.$$

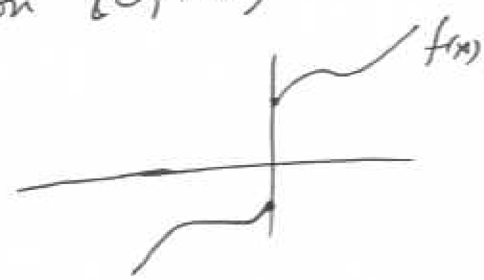
Fourier Sine and Cosine transforms

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx, \qquad \gamma \text{ is arbitrary.}$$

$x \geq 0$ $\quad$ $f(x)$ defined on $[0,+\infty)$

Introd. odd extension



$\boxed{\text{If } f(x) \text{ is odd}}$

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

$$= \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)\,[\cos\omega x + i\sin\omega x]\,dx = \frac{\gamma}{2\pi}\Bigg[\cancel{\int_{-\infty}^{\infty} \underbrace{f(x)\cos\omega x}_{\text{odd}}\cdot dx}^{\,0}$$

$$+\, i\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{i\gamma}{2\pi}\int_{-\infty}^{\infty} \overbrace{f(x)\sin\omega x}^{\text{odd}\times\text{odd} = \text{even}}\,dx$$

Thus,

$$\boxed{F(\omega) = \frac{2i\gamma}{2\pi}\int_0^\infty f(x)\sin\omega x\,dx} \quad (*)$$

From equ. (*), it's clearly seen that $F(\omega)$ is also and odd. function of ω.

Similarly,

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} \underbrace{F(\omega)\,(\cos(\omega x)}_{odd} - i\sin(\omega x))\,d\omega =$$

where $\cos(\omega x) - i\sin(\omega x) = e^{-i\omega x}$

$$= -\frac{2i}{\gamma}\int_{0}^{\infty} \underbrace{F(\omega)\sin(\omega x)}_{even}\,d\omega$$

By defining $\quad -\frac{2i}{\gamma} = 1$

we have

$$\boxed{\begin{aligned} f(x) &= \int_{0}^{\infty} F(\omega)\sin(\omega x)\,dx \\ F(\omega) &= \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin(\omega x)\,dx. \end{aligned}}$$

Similarly, if <u>$f(x)$ is even function</u>

$$f(x) = \int_{0}^{\infty} F(\omega)\cos\omega x\,dx$$

$$F(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos\omega x\,dx$$

Transform of Derivatives

Notation

$$C[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$ Fourier Cosine Transform of $f(x)$.

$$S[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx.$$ Fourier sine transform of $f(x)$.

$$C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty \frac{df}{dx}\cos\omega x\,dx \overset{I.P.P.}{=}$$

$$= \frac{2}{\pi}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x\,dx\right].$$

$$= -\frac{2}{\pi}f(0) + \omega\,\frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

$\therefore$

$$\boxed{C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f(x)]}$$

Similarly, $$\boxed{S\left[\frac{df}{dx}\right] = -\omega C[f]}$$

Assuming $f(x) \underset{x\to\infty}{\to} 0$

Comment: 1st derivatives in space may be a problem

$$C\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) + \omega\, S\left[\frac{df}{dx}\right] =$$

$$= -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 C[f]$$

$$S\left[\frac{d^2f}{dx^2}\right] = -\omega\, C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\omega f(0) - \omega^2 S[f].$$

Application:

From previous formulas, we realize that

a) Cosine F.T. requires a B.C. in $\frac{df}{dx}(0)$ "Derivative". Neumann cond.

b) Sine F.T. " " " " $f(0)$ Dirichlet B.C.

Application to a Semi-infinite BVP for heat equation.

$$\begin{cases} u_t = k\, u_{xx}, & 0 < x < \infty, \quad t > 0 \\ u(0,t) = g(t) \\ u(x,0) = f(x) \end{cases}$$

Notice that we have a nonhomog B.C. if $g(t) \neq 0$.

Since B.C. is of Dirichlet-type, we will use Fourier sine transform.

$$S[u(x,t)] = \bar{U}(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\sin \omega x \, dx$$

Applying Sine F.T. to the Heat Equ., we obtain

$$\frac{\partial \bar{U}}{\partial t} = k\left(\frac{2}{\pi}\omega \underbrace{u(0,t)}_{g(t)} - \omega^2 \bar{U}(\omega,t)\right) \quad (30.1)$$

From the I.C.: First-order linear nonhomog. ODE.

$$\bar{U}(\omega,0) = \frac{2}{\pi}\int_0^\infty u(x,0)\sin \omega x \, dx = \frac{2}{\pi}\int_0^\infty f(x)\sin \omega x \, dx \quad (30.2)$$

It can be solved using an integrating factor.

A particular case of (30.1) is if $g(t) \equiv 0$.

(homogeneous case).

Gen. Soln. of 1st order linear homog. ODE

$$\boxed{\bar{U}(\omega,t) = C(\omega)\, e^{-k\omega^2 t}} \quad (30.3)$$

Using I.C., $\bar{U}(\omega,0) = C(\omega)$ Combining with (30.2)

$$\boxed{C(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin \omega x \, dx} \quad (30.4)$$

Don't follow this instead follow Convolution Thm. application

from (30.3) taking the inverse F.T.

$$u(x,t) = \int_0^\infty \bar{v}(\omega,t) \sin \omega x \, d\omega =$$

$$\therefore \boxed{u(x,t) = \int_0^\infty c(\omega)\, e^{-k\omega^2 t} \sin \omega x \, d\omega} \quad (31.1)$$

Equations (31.1) and (30.4) form the Soln. of our BVP.

It's Same soln. obtained from sep. Vars.

A little more algebra will allow us to express this soln as

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty f(\bar{x}) \left[e^{-(x-\bar{x})^2/4kt} - e^{-(x+\bar{x})^2/4kt} \right] d\bar{x}$$

In fact, from (31.1) and using that $c(\omega)$ is an odd fn.

$$u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{c(\omega)\, e^{-k\omega^2 t} \sin \omega x}_{\text{even}} \, dx. = \int_{-\infty}^{\infty} \frac{c(\omega)}{2i} e^{-k\omega^2 t} e^{i\omega x} d\omega$$

$e^{i\omega x} = \cos\omega x + i \sin \omega x$

$$\int_{-\infty}^{\infty} \overset{\text{odd}}{c(\omega)\cos\omega x} = 0$$

$$\therefore \boxed{u(x,t) = \int_{-\infty}^{\infty} \frac{c(\omega)}{2i} e^{-k\omega^2 t} e^{i\omega x} d\omega} \quad (**)$$

$$\sin \omega x = \frac{e^{i\omega x} - \cos \omega x}{i}$$

On the other hand, if $\hat{f}(x) = \begin{cases} f(x), & x > 0 \\ -f(-x), & x < 0 \end{cases}$

odd extension ~~of~~ $f(x)$.

then

$$\frac{C(\omega)}{2i} = \frac{2}{\pi}\int_0^\infty f(x)\, \frac{\sin \omega x}{2i}\, dx =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}(x)\, \frac{\sin \omega x}{2i}\, dx. \qquad \int_{-\infty}^{\infty} \overset{\text{even}}{=} 2\int_0^\infty$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)\, \frac{\sin \omega x}{i}\, dx. \quad \underline{\underline{(*)}}$$

$$= \frac{1}{2\pi}\left[\cancel{\int_{-\infty}^{\infty} \hat{f}(x) \cos \omega x\, dx}^{\,0} + \int_{-\infty}^{\infty} \hat{f}(x)\, e^{-i\omega x}\, dx\right]$$

$$\Rightarrow \boxed{\frac{C(\omega)}{2i} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)\, e^{-i\omega x}\, dx} \quad (***)$$

$(**)$ and $(***)$ $\Rightarrow$

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} \hat{f}(\bar{x})\, e^{-(x-\bar{x})^2/4\kappa t}\, d\bar{x}$$

$(*)$ $\dfrac{\sin \omega x}{i} = ?$ $\qquad e^{-i\omega x} = \cos(-\omega x) - i \sin(\omega x)$

$$= \cos(\omega x) + \frac{\sin(\omega x)}{i}$$

$$= + e^{-i\omega x} \qquad = -\cos \omega x + \frac{\sin \omega x}{i}$$

Last HWK

$$\hat{f}(x) = \begin{cases} f(x), & x > 0 \\ -f(-x), & x < 0 \end{cases}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{\sqrt{4\pi \kappa t}} \int_0^\infty f(\bar{x}) \left[e^{-(x-\bar{x})^2/4\kappa t} - e^{-(x+\bar{x})^2/4\kappa t} \right] d\bar{x}.$$

Hwk problems:

R. Haberman: 10.5.1, 10.5.2, 10.5.6 - 10.5.8, 10.5.11, 10.5.14.

10.3.5, 10.3.7, 10.3.11, 10.3.18.