Complex form of Fourier Series.

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \right]}$$

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,} \qquad \boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx.}$$

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \begin{aligned} \cos\theta &= (e^{i\theta} + e^{-i\theta})/2 \\ \sin\theta &= (e^{i\theta} - e^{-i\theta})/2i \end{aligned}$$

$$\theta_n = \frac{n\pi}{L}$$

$$\Rightarrow f(x) \sim a_0 + \sum_{n=1}^{\infty} \left\{ a_n \left[\frac{e^{i\theta_n x} + e^{-i\theta_n x}}{2} \right] + b_n \left[\frac{e^{i\theta_n x} - e^{-i\theta_n x}}{2i} \right] \right\}$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} \left[(a_n - ib_n)\, e^{i\theta_n x} + (a_n + ib_n)\, e^{-i\theta_n x} \right]$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n - ib_n)\, e^{i\theta_n x} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n + ib_n)\, e^{-i\theta_n x} =$$

$m = -n$

$$= a_0 + \frac{1}{2}\sum_{m=-1}^{-\infty} (a_{-m} - ib_{-m})\, e^{i\theta_m x} + \frac{1}{2}\sum_{n=1}^{\infty} (\;)$$

$$a_{-m} \overset{\text{even}}{=} a_m, \qquad b_{-m} \overset{\text{odd}}{=} -b_m, \qquad \theta_{-m} = -\theta_m$$

$$= a_0 + \frac{1}{2} \sum_{\substack{m=-\infty \\ \downarrow \\ n}}^{-1} (a_m + i b_m) e^{-\theta_m x} + \sum_{m=1}^{\infty} (a_m + i b_m) e^{-\theta_m x}$$

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\theta_n x}, \qquad C_0 = a_0, \quad C_n = \frac{a_n + i b_n}{2}, \quad n \neq 0}$$

Similarly, if we change to keep : $e^{i\theta_n x}$.

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - i b_n) e^{i\theta_n x} + \frac{1}{2} \sum_{m=-1}^{-\infty} \underbrace{(a_{-m} + i b_{-m})}_{a_m - i b_m} e^{+i\theta_m x}$$

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n e^{i\theta_n x}, \qquad \bar{C}_0 = a_0, \quad \bar{C}_n = \frac{a_n - i b_n}{2}, \; n \neq 0}$$

If $L = \pi$,

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-inx} \quad \text{or} \quad f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n e^{inx}}$$

Determination of the coefficients C_n and $\bar{C}_n$ in terms of the function $f(x)$.

(I) If $n \neq 0$ $\quad C_n = \dfrac{a_n + ib_n}{2} =$

$n = \ldots -2, -1, 1, 2, \ldots$

$$= \frac{1}{2L}\left[\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx + i\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right] =$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{i\frac{n\pi}{L}x}\,dx$$

for $n = 0$ $\quad C_0 = a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx = \dfrac{1}{2L}\int_{-L}^{L} f(x)\,e^{i0x}\,dx$

(II) If $n \neq 0$ $\quad \bar{C}_n = \dfrac{a_n - ib_n}{2} =$

$$= \frac{1}{2L}\left[\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx - i\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-i\frac{n\pi}{L}x}\,dx.$$

Summarizing: Complex representation of Fourier Series.

(I)

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\frac{n\pi}{L}x}} \tag{3.1}$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\,dx.} \tag{3.2}$$

or

(II)

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{L}x}} \tag{3.3}$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx} \tag{3.4}$$

Transition from a discrete set of eigenvalues to a Continuous set.

Substitution of (3.2) into (3.1) leads to

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\frac{n\pi}{L}x} \Rightarrow$$

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} f(\bar{x}) e^{i\frac{n\pi}{L}\bar{x}} d\bar{x} \right] e^{-i\frac{n\pi}{L}x}} \quad (3.5)$$

Fourier series identity.

Calling $\omega = \frac{n\pi}{L}$ and $\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$n = \dots -2, -1, 0, 1, 2, \dots$ $\Rightarrow \frac{1}{2L} = \frac{\Delta\omega}{2\pi}$

Subst. into (3.5)

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{-L}^{L} f(\bar{x}) e^{i\omega\bar{x}} d\bar{x} \right] e^{-i\omega x} \quad (3.6)$$

when $L \to \infty$, the distance between consecutive, $\omega = \sqrt{\lambda}$, $\Delta\omega \to 0 \Rightarrow$ that all real numbers are possible ω.

Therefore, when $L \to \infty$ equ. (3.6) can be written

as
$$\boxed{f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}\right] e^{-i\omega x}\, d\omega} \quad (3.7)$$

"Fourier integral identity."

We claim that the two improper integrals in the (3.7) expression converge

if $f(x)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

For example, if $f(x)$ is piecewise smooth and $f(x) \xrightarrow[x \to \pm\infty]{} 0$

sufficiently fast, then $f(x)$ is absolutely integrable.

Now, we are ready for our definition:

Def- The function

$$\boxed{F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}}$$

is called the Fourier Transform of $f(x)$.

and equ. (3.7) can be written as

$$\boxed{f(x) \sim \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega}$$ Inverse Fourier transform of $F(\omega)$

Comparison

For $f(x)$ piecewise Smooth on $[-L,L]$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{-\frac{in\pi}{L}x}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi}{L}x}\, dx.$$

Discrete Fourier Series of $f(x)$ on $[-L,L]$.

or

$$f(x) = \sum_{n=-\infty}^{\infty} \bar{C}_n \, e^{\frac{in\pi}{L}x}$$

$$\bar{C}_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi}{L}x}\, dx$$

For $f(x)$ absolutely integrable on $(-\infty,\infty)$.

$$f(x) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, dx$$

Inverse Fourier transform.

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}$$

Fourier transform Of $f(x)$

$$f(x) = \int_{-\infty}^{\infty} C(\omega)\, e^{i\omega x}\, dx$$

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{-i\omega\bar{x}}\, d\bar{x}$$

Application to Heat Equation on an infinite domain.

$$\begin{cases} u_t = K u_{xx}, & -\infty < x < \infty, \quad t > 0 \quad (5.0) \\ u(x,0) = f(x) & (5.0.1) \\ + \text{B.C's at } \infty : \; u(-\infty, t) = 0, & u(+\infty, t) = 0 \quad (5.0.2) \end{cases}$$

Separation of variables leads to

$$u(x,t) = \phi(x)\, h(t)$$

(I) $\dfrac{dh}{dt} + \lambda K h = 0.$

(II) EVP:
$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda \phi = 0 & (5.1) \\ |\phi(-\infty)| < \infty, \quad |\phi(+\infty)| < \infty & (5.2) \end{cases}$$

Why not $\phi(\infty) = 0$, $\phi(-\infty) = 0$. ?

In order to satisfy (5.1) and (5.2) $\lambda \geq 0$

(If $\lambda < 0$, $\phi(x)$ would be a combination of exponentials and won't be bounded as $x \to \infty$ and $x \to -\infty$ at the same time.)

Thus, $\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

Satisfies both (5.1) and (5.2) for all $\lambda \geq 0$.

All $\lambda \geq 0$ are eigenvalues of (5.1) and (5.2)

"Continuous spectrum".

Now, by extending the superposition principle from the discrete to the continuous case, we obtain

$$u(x,t) = \int_0^\infty \left[C_1(\lambda) \cos(\sqrt{\lambda}\, x)\, e^{-\lambda k t} + C_2(\lambda) \sin(\sqrt{\lambda}\, x)\, e^{-\lambda k t} \right] d\lambda$$

using $\lambda = \omega^2$

$$u(x,t) = \int_0^\infty \left[A(\omega) \cos(\omega x)\, e^{-k\omega^2 t} + B(\omega) \sin(\omega x)\, e^{-k\omega^2 t} \right] d\omega$$

or introducing complex functions.

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, e^{-k\omega^2 t}\, d\omega} \qquad (6.1)$$

where $C(\omega) = \begin{cases} \dfrac{A(-\omega) - iB(-\omega)}{2}, & \omega < 0 \\ \dfrac{A(\omega) + iB(\omega)}{2}, & \omega > 0 \end{cases}$

See next two pages

To satisfy I.C.

$$\boxed{u(x,0) = f(x) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, d\omega} \qquad (6.2)$$

If $f(x)$ is absolutely integrable, then (6.2) is the inverse Fourier transform of $f(x)$ and $C(\omega)$ is its Fourier transform

From the previous definition,

$$\boxed{C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega \bar{x}}\, d\bar{x}} \qquad (6.3)$$

Therefore, the Solution of the Heat conduction equation on an infinite domain with BC's: $u(\pm\infty,t)=0$. is given by the pair:

$$\begin{cases} u(x,t) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, e^{-K\omega^2 t}\, d\omega & (7.1) \\ C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x} & (7.2) \end{cases}$$

It's possible to combine these two into a single integral representation:

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\bar{x})\,\frac{1}{\sqrt{4\pi K t}}\, e^{-(x-\bar{x})^2/4Kt}\, d\bar{x}}$$

In fact, Substitution of (7.2) into (7.1) leads to

$$\boxed{\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}\right] e^{-i\omega x}\, e^{-K\omega^2 t}\, d\omega \\ \text{interchanging order of integration} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\left[\int_{-\infty}^{\infty} e^{-K\omega^2 t}\, e^{-i\omega(x-\bar{x})}\, d\omega\right] d\bar{x} \end{aligned}} \quad (7.3)$$

The expression in brackets

$$\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})} d\omega = g(x-\bar{x})$$

where $\boxed{g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega}$

is the inverse Fourier transform of $e^{-k\omega^2 t}$

Inverse Fourier Transform of a Gaussian

Consider $\hat{G}(\omega) = e^{-\alpha\omega^2}$

and its inverse F.T.

$$\hat{g}(x) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega$$

we will prove first that

$$\boxed{\hat{g}'(x) = -\frac{x}{2\alpha}\hat{g}(x)}$$ 1st order linear ODE.

In fact,

$$\hat{g}'(x) = \int_{-\infty}^{\infty} -i\omega\, e^{-\alpha\omega^2} e^{-i\omega x} d\omega =$$

$$= \int_{-\infty}^{\infty} \frac{-i\omega}{2\alpha} 2\alpha\, e^{-\alpha\omega^2} e^{-i\omega x} d\omega = \frac{i}{2\alpha}\int_{-\infty}^{\infty} \frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right) e^{-i\omega x} d\omega$$

Integration by parts

$$\hat{g}'(x) = \frac{i}{2\alpha}\left[\cancel{e^{-\alpha\omega^2} e^{-i\omega x}}^{\,0}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-ix)\, e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega \right]$$

$$= -\frac{x}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega = \frac{-x}{2\alpha}\,\hat{g}(x) \checkmark$$

Solving this ODE:

$$\boxed{\hat{g}(x) = C\, e^{x^2/4\alpha} = \hat{g}(0)\, e^{x^2/4\alpha}.}$$

Where $\hat{g}(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2}\, d\omega$

Calling: $z^2 = \alpha\omega^2 \Rightarrow \omega = \frac{z}{\sqrt{\alpha}} \Rightarrow d\omega = \frac{1}{\sqrt{\alpha}}\, dz$

$$\Rightarrow \boxed{\hat{g}(0) = \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2}\, dz = }$$

Since

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy} =$$

$$= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy\right]^{1/2} \underset{\text{Using polar coords}}{=}$$

$$= \left[\int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr\right]^{1/2} = \left[2\pi\int_0^{\infty} r e^{-r^2}\, dr\right]^{1/2} = \left[-\pi\int_0^{\infty} -2r\, e^{-r^2}\, dr\right]^{1/2} =$$

$$= \left[-\pi\, e^{-r^2}\Big|_0^{\infty}\right]^{1/2} = (\pi)^{1/2} \checkmark$$

Finally,

If $\boxed{\hat{G}(\omega) = e^{-\alpha\omega^2}}$ its inverse F.T. is

given by

$$\boxed{\hat{g}(x) = \sqrt{\frac{\pi}{\alpha}}\; e^{-x^2/4\alpha}}$$

Therefore,

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x}\, d\omega \overset{\alpha = kt}{=} \sqrt{\frac{\pi}{kt}}\; e^{-\frac{x^2}{4kt}}$$

Substitution into (7.3)

$$u(x,t) = \frac{1}{2\pi} \frac{\sqrt{\pi}}{\sqrt{kt}} \int_{-\infty}^{\infty} f(\bar{x})\; e^{-\frac{(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

or

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\bar{x})^2}{4kt}} f(\bar{x})\, d\bar{x}.}$$