Sturm-Liouville Eigenvalue Problem.

Consider the Eigenvalue problem:

$$\begin{cases} -\nabla\cdot\left(p(\vec{x})\nabla u\right) + q(\vec{x})u = \lambda\,\omega(\vec{x})\,u, & \vec{x}\in\Omega \quad (1.1) \\ a\,\phi(\vec{x}) + b\,\nabla\phi\cdot\hat{n}(\vec{x}) = 0, & \vec{x}\in\partial\Omega \quad (1.2) \end{cases}$$

where $\omega(\vec{x})>0$, $p(\vec{x})>0$, $q, p,$ and ω are conts on $\bar{\Omega}$. Also, $p\in C^1(\Omega)$

<u>Thm.</u>- The EVP (1.1) and (1.2) Satisfies:

i) The eigenvalues are real.

ii) There are infinitely many eigenvalues

$$0<\lambda_1\le\lambda_2\le\ldots \quad \text{with } \lambda_n \xrightarrow[n\to\infty]{} \infty$$

iii) Eigenfns Corresponding to different eigenvalues are orthogonal with respect to the weight function $\omega(\vec{x})$.

$$\langle\phi_i,\phi_j\rangle \equiv \int_a^b \phi_i(\vec{x})\,\phi_j(\vec{x})\,\omega(\vec{x})\,d\vec{x} = 0, \quad i\ne j.$$

iv) The set of eigenfunctions $\{\phi_i(\vec{x})\}_{i=1}^{\infty}$ form a complete set in $L^2(\Omega)$ $\left(\int_\Omega |f(\vec{x})|^2 w(\vec{x}) d\vec{x} < \infty \right)$

Any $f \in L^2(\Omega)$ can be written as

$$f(\vec{x}) = \sum_{i=1}^{N} C_i \, \phi_i(\vec{x})$$

where
$$C_i \equiv \frac{\langle f, \phi_i \rangle}{\|\phi_i\|^2} = \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}$$

and
$$\int_\Omega \left[f(\vec{x}) - \sum_{i=1}^{N} C_i \phi_i(\vec{x}) \right]^2 w(\vec{x}) \, d\vec{x} \xrightarrow[N\to\infty]{} 0$$

Convergence in the mean-square sense or Convergence in $L^2(\Omega)$ or mean-square Convergence.

<u>Thm 6.23</u>. The BVP:

$$\begin{cases} -\nabla \cdot (p(\vec{x}) \nabla u) + q \, u = \mu w(\vec{x}) u + f(\vec{x}) & (2.1) \\ a \, \phi(\vec{x}) + b \, \nabla\phi \cdot \hat{n}(\vec{x}) = 0 & (2.2) \end{cases}$$

Satisfies the following:

i) If μ is not an eigenvalue of the corresponding EVP ($f \equiv 0$), then ① there is a ② unique soln. for all functions $f \in L^2(\Omega)$ with weight $w(\vec{x})$.

ii) If $\mu = \lambda_j$ (eigenvalue with eigenfunction ϕ_j) and $f \perp \phi_j$, then (2.1)- (2.2) has infinitely many solutions

iii) If $\mu = \lambda_j$ and $f \not\perp \phi_j$, then (2.1) - (2.2) has no solutions.

FREDHOLM ALTERNATIVE for Linear Systems (algebraic)

Consider the nonhomogeneous linear algebraic system

$$A\hat{u} = \lambda \hat{u} + \vec{f}, \qquad \vec{f} \not\equiv \vec{0} \qquad (1)$$

where $A_{n\times n}$ is real and symmetric, $\vec{f} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Thm. The linear system (1) satisfies:

i) If λ is not an eigenvalue of the corresponding homogeneous system:

$$A\hat{u} - \lambda \hat{u} = 0, \qquad (2)$$

then the linear system (1) has a unique soln. $u \in \mathbb{R}^n$, for all given vectors $\vec{f}$ in $\mathbb{R}^n$.

ii) If $\lambda = \lambda_j$ is an eigenvalue of (2) and $\vec{f} \perp \vec{e}^{\,j}$, where $\vec{e}^{\,j}$ is an eigenvector corresp. to λ_j, then (1) has infinitely many solutions.

iii) If $\lambda = \lambda_j$ and $f \not\perp \vec{e}^{\,j}$, then (1) has no solutions.

Proof.– Since A is symmetric, then there exists an orthonormal basis $\{\vec{e}^{\,i}\}_{i=1}^{n}$ of eigenvectors of A.

Therefore, any soln. vector $\hat{u}$ can be written as

$$\hat{u} = \sum_{i=1}^{n} c_i \vec{e}^{\,i} \qquad (3)$$

also,

$$\hat{f} = \sum_{i=1}^{n} f_i \vec{e}^{\,i} \qquad (4)$$

Subst. into (1) leads to

$$A\left(\sum_{i=1}^{n} c_i \vec{e}^{\,i}\right) - \lambda\left(\sum_{i=1}^{n} c_i \vec{e}^{\,i}\right) = \sum_{i=1}^{n} f_i \vec{e}^{\,i}$$

$$\Rightarrow \sum_{i=1}^{n} c_i A \hat{e}^{i} - \sum_{i=1}^{n} c_i \lambda \vec{e}^{\,i} = \sum_{i=1}^{n} f_i \vec{e}^{\,i}$$

$$\Rightarrow \sum_{i=1}^{n} \left[c_i \lambda_i - c_i \lambda - f_i\right] \vec{e}^{\,i} = \vec{0}$$

where λ_i is the corresponding eigenvalue for $\vec{e}^{\,i}$, $i = 1, .., n$.

Using orthog. yields

$$\boxed{c_i(\lambda_i - \lambda) = f_i} \qquad i = 1, 2, .., n \qquad (5)$$

Thus, $\vec{u} = \sum_{i=1}^{n} c_i \vec{e}^{\,i}$ is a soln. of (1)

if and only if

$$C_i(\lambda_i - \lambda) = f_i, \qquad i=1,2,.., n. \qquad (3.1)$$

There are three possibilities:

i) $\lambda \neq \lambda_i$, $i=1,2,..,n$

then $C_i = \dfrac{f_i}{\lambda_i - \lambda}$, $i=1,2,..,n$ (3.2)

Verify that $\vec{u}$ defined with this c_i's satisfies (1)

and there is a soln $\vec{u}$ given by (1). This soln. is unique due to the arbitrariness of $\vec{u}$ and the uniqueness of the representation of $\vec{u}$ in the basis $\{\vec{e}^{\,i}\}_1^n$.

ii) $\lambda = \lambda_j$, for certain j.

In this case, there is a solution if $\vec{f} \perp \vec{e}^{\,j}$ (or $f_j = 0$)

$$\left(\vec{f} = \sum_{i=1}^{n} f_i \vec{e}^{\,i} \Rightarrow \vec{f}\cdot\vec{e}^{\,j} = \sum_{i=1}^{n} f_i (\vec{e}^{\,i}\cdot\vec{e}^{\,j}) = f_j \right.$$

$$\left. \text{thus} \quad \vec{f}\cdot\vec{e}^{\,j} = 0 \Leftrightarrow f_j = 0 \right)$$

In fact, From (5) $\quad C_i = \dfrac{f_i}{\lambda_i - \lambda_j}$, $\quad i \neq j$

and $\quad c_j(0) = 0 \quad \Rightarrow c_j$ is arbitrary

Any $\vec{u} = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \vec{e}^{\,i} + d\,\vec{e}^{\,j}$, $d \in \mathbb{R}$ (arbitrary)

Where $c_i = \dfrac{f_i}{\lambda_i - \lambda_j}$ will be a solution of (1).

<u>Verification:</u>

$$(A - \lambda_j I)\vec{u} = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i (A - \lambda_j I)\vec{e}^{\,i} + d(A - \lambda_j I)\vec{e}^{\,j}$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{f_i}{\cancel{\lambda_i - \lambda_j}} \cancel{(\lambda_i - \lambda_j)}\, \vec{e}^{\,i} + d\,(0) = \sum_{\substack{i=1 \\ i \neq j}}^{n} f_i \vec{e}^{\,i} = \vec{f}$$

because $f_j = 0$.

iii) If $\lambda = \lambda_j$, for certain j and $\vec{f} \not\perp \vec{e}^{\,j}$ (or $f_j \neq 0$)

For any vector $\vec{u} = \sum_{i=1}^{n} c_i \vec{e}^{\,i}$ to satisfy (1) (recall $\{\vec{e}^{\,i}\}$ form a basis)

the condition $c_j \underbrace{(\lambda_j - \lambda_j)}_{=0} = \underbrace{f_j}_{\neq 0}$ should be verified

There is obviously no c_j value to satisfy this last equation. Therefore, the system (1) has no solution in this case.