Fredholm Integral Equations with Symmetric Kernels.

Def. – Consider
$$Ku \equiv \int_a^b K(x,y)\, u(y)\, dy, \qquad (1)$$

where $K(x,y)$ is a real-valued function on $\underbrace{[a,b]\times[a,b]}_{=R}$

If $K(x,y) = K(y,x)$, for all $(x,y)\in R$

we will say that the kernel $K(x,y)$ is real and symmetric.

Importance of this can be illuminated by considering Sturm-Liouville differential operator and regular S-L EVP.

$$Lu \equiv (p(x)\, u')' + qu$$

we prove that for regular S-L EVP
$$\begin{cases} Lu + \lambda\sigma(x) u = 0 \\ (\beta_1 u + \beta_2 u')(a) = 0 \\ (\beta_3 u + \beta_4 u')(b) = 0 \end{cases}$$

$$\int_a^b [uLv - vLu]\, dx = 0 \qquad \text{(self-adjoint operator)}$$

$$p(x)\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\Bigg|_a^b$$

$$\int_a^b uLv\, dx = \int vLu\, dx$$

or
$$(u, Lv) = (v, Lu).$$

Using this result we prove that all eigenvalues are real and that eigenfunctions corresponding to different eigenvalues are orthogonal.

Exactly the same result can be proved for Fredholm integral operator with symmetric kernels.

Lemma 1.2.- If $k(x,y)$ defined in $R \equiv [a,b] \times [a,b]$ is

a) real, b) continuous on R, c) symmetric

then the operator K has the symmetry property

$$(Ku, v) = (v, Ku).$$

Proof.-

$$(Ku, v) = \left(\int_a^b k(x,y)\, u(y)\, dy,\ v(x) \right) =$$

$$= \int_a^b \left[\int_a^b k(x,y)\, u(y)\, dy \right] \overline{v(x)}\, dx =$$

$$= \int_a^b \int_a^b k(x,y)\, u(y)\, \overline{v(x)}\, dy\, dx$$

Renaming variables $x \leftrightarrow y$.

$$(Ku, v) = \int_a^b \int_a^b k(y,x)\, u(x)\, \overline{v(y)}\, dx\, dy$$

Interchanging order of integration

$$\int_a^b \int_a^b k(y,x)\, u(x)\, \overline{v(y)}\, dy\, dx$$

$$= \int_a^b u(x) \left[\int_a^b k(y,x)\, \overline{v(y)}\, dy \right] dx$$

$$= \int_a^b u(x)\, \overline{\left(\int_a^b k(y,x)\, v(y)\, dy \right)}\, dx$$

Symmetric

$$= \int_a^b u(x)\, \overline{\left(\int_a^b k(x,y)\, v(y)\, dy \right)}\, dx = (u, Kv)$$

or $$\boxed{(Ku, v) = (u, Kv)} \qquad (3)$$

<u>Def.</u>- If an integral operator K satisfies (3), then K is called integral Symmetric operator.

Thm 1.5
1.6

Consider the integral operator K defined by (1), with kernel $K(x,y)$ satisfying properties (a)–(c) of previous lemma then the following statements are true

1.- If $K(x,y)$ is not separable, the operator K has infinitely many eigenvalues: $\mu_1, \dots, \mu_n, \dots$ with finite multiplicity. They can be ordered

$$0 \leq \dots \leq |\mu_n| \leq \dots \leq |\mu_2| \leq |\mu_1|.$$

and $\lim_{n \to \infty} \mu_n = 0$.

2.- All the eigenvalues are real.

3.- Eigenfunctions corresponding to different eigenvalues are orthogonal.

4.- Any $f \in L_2[a,b]$ (square integrable function) can be expanded in terms of the set of orthonormal eigenfunctions $\{\phi_k(x)\}_{k=1}^{\infty}$ as

$$f(x) = \sum_{k=1}^{\infty} f_k \, \phi_k(x), \quad \text{where } f_k \equiv (f, \phi_k) \quad k = 1, 2, \dots$$

and $S_n = \sum_{k=1}^{n} f_k \, \phi_k(x) \longrightarrow f$ in $L_2[a,b]$.

Proof.- 2.- All eigenvalues are real.

Let λ be an eigenvalue of K, then there exists $\hat{u}(x) \equiv 0$ such that $Ku = \lambda u$

multiply by $\overline{u(x)}$ on the right and $\int_a^b dx$

$$\int_a^b (Ku)(x)\,\overline{u(x)}\,dx = \int_a^b \lambda u(x)\overline{u(x)}\,dx \Leftrightarrow \boxed{(Ku,u) = \lambda(u,u)} \quad (5.1)$$

multiply $\overline{(Ku)}(x)$ and $\overline{\lambda u}$ by $u(x)$ on the left and $\int_a^b dx$

$$\int_a^b u(x)\,(\overline{Ku})(x)\,dx = \int_a^b u(x).\overline{\lambda u(x)}\,dx \Leftrightarrow (u,Ku) =$$

Subtracting (5.2) from (5.1)

$$(Ku,u) - (u,Ku) = (\lambda u,u) - (u,\lambda u)$$

$$0 = \lambda(u,u) - \bar{\lambda}(u,u) = (\lambda - \bar{\lambda})(u,u)$$

but $(u,u) = \|u(x)\|^2 \neq 0$, then $\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$

or λ real.

Rmk: $(u,\lambda u) = \int_a^b u(x)\,\overline{\lambda u(x)}\,dx = \int_a^b \bar{\lambda}\, u(x)\overline{u(x)}\,dx = \bar{\lambda}\,(u,u)$.

3.- Eigenfunctions corresponding to different eigenvalues are orthogonal.

<u>Proof:-</u>

Let λ and ν be two different eigenvalues of K, then there exist $u(x) \neq 0$ and $v(x) \neq 0$, respectively, such that

$$\boxed{Ku = \lambda u,} \qquad (6.1)$$

$$Kv = \nu v. \Rightarrow \boxed{\overline{Kv} = \overline{\nu v} \qquad (6.2)}$$

Multiplying (6.1) by $v(x)$ on the right and $\int_a^b dx$

$$(Ku, v) = (\lambda u, v). \qquad (6.3)$$

Multiplying (6.2) by $u(x)$ on the left and $\int_a^b dx$

$$(u, Kv) = (u, \nu v) \qquad (6.4)$$

Subtracting (6.4) from (6.3).

$$0 \overset{\text{Hypothesis}}{=} (Ku, v) - (u, Kv) = (\lambda u, v) - (u, \nu v) = \lambda (u, v) - \bar{\nu}(u, v)$$

$$= (\lambda - \bar{\nu})(u, v)$$

Then $(\lambda - \bar{\nu})(u, v) = 0$, but $\lambda \neq \bar{\nu}$

$$\Rightarrow (u, v) = 0 \quad \text{or} \quad u(x) \perp v(x) \quad \checkmark$$

The importance of theorems 1.5 and 1.6 to solve the integral equation

$$Ku - \lambda u = f \qquad (7.1)$$

where K is real, continuous and symmetric and f is continuous on $[a,b]$, is that if we happen to know the eigenfunctions $\{\phi_k\}_1^n$ and corresponding eigenvalues $\{\alpha_k\}_{k=1}^{\infty}$, the we can represent

$$\boxed{u(x) = \sum_{k=1}^{\infty} u_k \, \phi_k(x),}$$

and easily obtain : $\boxed{u_k = \dfrac{f_k}{\alpha_k - \lambda}}$, $(\text{if } \lambda \neq \mu_k)$ $k=1,2,\ldots$

where $f_k \equiv (f, \phi_k)$.

In fact, let's represent

$$f(x) = \sum_{k=1}^{\infty} f_k \phi_k(x), \qquad u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x) \qquad (7.2)$$

Subst. in (7.2) leads to

$$K\left(\sum_{k=1}^{\infty} u_k \phi_k(x)\right) - \lambda\left(\sum_{k=1}^{\infty} u_k \phi_k(x)\right) = \sum_{k=1}^{\infty} f_k \phi_k(x) \qquad (7.3)$$

Now, $$K\left(\sum_{k=1}^{\infty} u_k \phi_k\right) = \sum_{k=1}^{\infty} u_k (K\phi_k) = \sum_{k=1}^{\infty} u_k (\alpha_k \phi_k)$$

Therefore, after Substitution on (7.3)
$$\sum_{k=1}^{\infty} \alpha_k \phi_k u_k - \sum_{k=1}^{\infty} \lambda u_k \phi_k = \sum_{k=1}^{\infty} f_k \phi_k$$

or $$\sum_{k=1}^{\infty} (\alpha_k u_k - \lambda u_k)\phi_k = \sum_{k=1}^{\infty} f_k \phi_k$$

Since the representation on $\{\phi_k\}_1^{\infty}$ is unique
$$\alpha_k u_k - \lambda u_k = f_k, \qquad \text{for all } k = 1, 2, \ldots$$

if $\lambda \neq \alpha_k$, then
for all $k=1,2,\ldots$ $$\boxed{u_k = \frac{f_k}{\alpha_k - \lambda}}$$

and $$\boxed{u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x)}$$

If for $k = m$, $\lambda = \alpha_m$, then
$$\alpha_m u - \lambda u_m = f_m$$
$\Rightarrow$ $0 = f_m$, then the only possibility of solution for our integral equation is if $f_m = (f, \phi_m) = 0$ or $f \perp \phi_m$, and in this case u_m is arbitrary. Therefore, we have infinitely many solutions represented by $\boxed{u(x) = u_m \phi_m(x) + \sum_{\substack{k=1 \\ k \neq m}}^{\infty} u_k \phi_k(x)}$