Green's function for BVP governed by 2nd order linear ODE.

Consider the BVP:

$$\begin{cases} u''(x) = f(x), & 0 < x < L, \quad f \in C[0,L]. \quad (1) \\ u(0) = 0, \quad u(L) = 0. & (2) \end{cases}$$

(I) Uniqueness: Consider the associated homogeneous problem

$$\begin{cases} w''(x) = 0 & (3) \\ w(0) = 0, \quad w(L) = 0 & (4) \end{cases}$$

Construction: $w(x) = Ax + B$ (integrating twice)

BC's. $\Rightarrow A = 0, \; B = 0 \Rightarrow w(x) \equiv 0.$

(II) Existence: Integrating $f(x)$ twice

$$u'(x) = \int^x f(x)dx + C_1 = F(x) + C_1$$

$$u(x) = \int F(x)dx + C_1 x + C_2 = G(x) + C_1 x + C_2$$

C_1 and C_2 are determined from BC's.

III Alternative for existence. Construction of solution

Solve two half BVP first:

a) $u''(x) = 0$

$u(0) = 0, \quad u(L) \neq 0$

$u_1(x) = Ax$, in particular $\boxed{u_1(x) \equiv x}$

b) $u''(x) = 0$

$u(0) \neq 0, \quad u(L) = 0$

$u_2(x) = -\frac{B}{L}x + B$, in part. $B=L$ $\boxed{u_2(x) = L - x}$

These two solutions are linearly independent since

$$W[u_1, u_2] = \begin{vmatrix} x & L-x \\ 1 & -1 \end{vmatrix} = -x - L + x = -L \neq 0$$

Turn

Therefore, the general solution of

$u''(x) = 0$

Can be written as

$$\boxed{u(x) = C_1 x + C_2 (L - x)}$$

$$\boxed{u(x) = C_1 u_1(x) + C_2 u_2(x)}$$

Rmk: Instead of $u_1(x) = x$, $u_2(x) = L - x$, we could use any two linearly independent soln. of $u''(x) = 0$ to construct the general soln:

$$\boxed{u(x) = C_1 u_1(x) + C_2 u_2(x)}$$

VARIATION OF PARAMETERS.

c) Finally, Solve the nonhomog. BVP. We will prove existence by Constructing the actual soln. for this BVP.

$$\begin{cases} u''(x) = f(x) & (3.1) \\ u(0)=0, \quad u(L) = 0 & (3.2) \end{cases}$$

Using Variation of parameters the general Solution of (3.1) is given by

$$u(x) = C_1 u_1(x) + C_2 u_2(x) + u_p(x).$$

where
$$u_p(x) = \left(-\int_0^x \frac{u_2(x_0)\, f(x_0)}{W[u_1,u_2]}\, dx_0\right) u_1(x) + \left(\int_0^x \frac{u_1(x_0)\, f(x_0)}{W[u_1,u_2]}\, dx_0\right) u_2(x)$$

In our particular case, the general Solution is given by

$$u(x) = C_1 x + C_2 (L-x) + \left(\frac{1}{L}\int_0^x f(x_0)\,(L-x_0)\, dx_0\right) x$$

$$+ \left(-\frac{1}{L}\int_0^x f(x_0)\, x_0\, dx_0\right)(L-x).$$

Turn

Using the BC's, to determine the unique soln. of BVP (3), we obtain

$$u(x) = -\frac{x}{L}\int_x^L f(x_0)\,(L-x_0)\, dx_0 - \frac{L-x}{L}\int_0^x f(x_0)\, x_0\, dx_0$$

$$= \int_0^L f(x_0)\, G(x,x_0)\, dx_0, \qquad G(x,x_0) = \begin{cases} -\dfrac{x_0(L-x)}{L}, & x_0 < x \\ -\dfrac{x(L-x_0)}{L}, & x_0 > x \end{cases}$$

IV Finding the solution using eigenfunction expansions.

Consider the eigenvalue problem associated to the original BVP.

$$\begin{cases} u''(x) = -\lambda u(x), & 0 < x < L & (5) \\ u(0) = 0, \quad u(L) = 0 & & (6) \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, \ldots$$

Next, expand $u(x)$ and $f(x)$ present in our initial ODE (1) in terms of eigenfns. $\{\phi_n(x)\}$.

$$u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x), \qquad f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

Subst. in orig. equ. (1)

$$\left(\sum_1^{\infty} u_n \phi_n(x)\right)'' = \sum_1^{\infty} f_n \phi_n(x)$$

$$\Rightarrow \quad \sum_1^{\infty} u_n \phi_n'' = \sum_1^{\infty} f_n \phi_n$$

$$\Rightarrow \quad \sum_1^{\infty} u_n(-\lambda_n)\phi_n = \sum_1^{\infty} f_n \phi_n \Rightarrow -u_n \lambda_n = f_n \Rightarrow \boxed{u_n = \frac{-f_n}{\lambda_n}}$$

or $$u_n = \frac{-f_n}{\left(\frac{n\pi}{L}\right)^2}, \quad n = 1, 2, \ldots$$

Therefore, solution of our BVP: $$\boxed{u(x) = -\sum_{n=1}^{\infty} \frac{L^2 f_n}{(n\pi)^2} \sin\left(\frac{n\pi}{L}x\right)}$$

Now,

$$f_n \equiv \frac{2}{L}\int_0^L f(x_0)\,\sin\frac{n\pi}{L}x_0\,dx_0$$

$$\therefore\quad U(x) = \frac{-2}{L}\int_0^L f(x_0)\left(\sum_{n=1}^{\infty}\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x_0\right)\sin\left(\frac{n\pi}{L}x\right)\right)dx_0$$

or

$$U(x) = \int_0^L f(x_0)\underbrace{\left(\frac{-2}{L}\sum_{n=1}^{\infty}\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x_0\right)\right)}_{\equiv\; G(x,x_0).}dx_0$$

or

$$U(x) = \int_0^L G(x,x_0)\,f(x_0)\,dx_0$$

where

$$G(x,x_0) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x_0\right)$$

Compare with

$$G(x,x_0) = \begin{cases} -\dfrac{x_0\,(L-x)}{L}, & x_0 < x \\[2ex] -\dfrac{x\,(L-x_0)}{L}, & x_0 > x. \end{cases}$$

Problem 5.2.6 Logan's book

Graphic: