Fundamental Solutions for Poisson's equ or Infinite space Green's function.

Consider Poisson's equation in the infinite space $\mathbb{R}^3$ or the infinite plane $\mathbb{R}^2$.

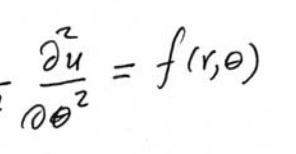
$$\begin{cases} \nabla^2 u(\vec{x}) = f(\vec{x}), & \vec{x} \in \mathbb{R}^2 \\ + \text{B.C. at } \infty \ (\text{to be determined}). & \\ \quad r \to \infty & \end{cases}$$

In polar coordinates:

$$\begin{cases} \nabla^2_{r,\theta} u(r,\theta) = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r,\theta) \\ + \text{B.C. when } r \to \infty. \qquad r \equiv \sqrt{x^2+y^2}. \end{cases}$$

The Corresponding BVP for the Green's fun. for a Source located at $\vec{x} = \vec{x}_0$, and $\vec{r} \equiv \sqrt{(x-x_0)^2+(y-y_0)^2}$.

$$\begin{cases} \nabla^2_{r,\theta} G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \mathbb{R}^2 \\ + \text{B.C. (homog.) when } r \to \infty & \end{cases}$$



Now, the response due to a point source at $\vec{x} = \vec{x}_0$ is symmetric in the radial direction $\Rightarrow \frac{\partial}{\partial \theta} = 0$.

Thus, the Green's fn Satisfies:

$$\begin{cases} \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), \quad \vec{x}\in\mathbb{R}^2 \\ + \text{B.C. at } r\to\infty. \end{cases}$$

If $\vec{x}\neq\vec{x}_0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \Rightarrow r\frac{dG}{dr} = C_1$$

$$\Rightarrow \frac{dG}{dr} = \frac{C_1}{r} \Rightarrow \boxed{G(\vec{x},\vec{x}_0) = C_1\ln(r) + C_2}, \quad \vec{x}\neq\vec{x}_0. \qquad (10.1)$$

The unknown constants can be obtained from an equivalent of the Jump Condition. This is

$$\iint_{S_\varepsilon(\vec{x}_0)} \nabla^2 G(\vec{x},\vec{x}_0)\, dS_{\vec{x}} = \iint_{S_\varepsilon(\vec{x}_0)} \delta(\vec{x}-\vec{x}_0)\, dS_{\vec{x}} = 1$$

$S_\varepsilon(\vec{x}_0) \hookrightarrow$ Small circle of radius ε.

$$\xrightarrow{\text{div thm}} \int_{C_\varepsilon} \underbrace{\nabla G(\vec{x},\vec{x}_0)\cdot\hat{n}(\vec{x})}_{\frac{\partial G}{\partial r}(\vec{x},\vec{x}_0)}\, dl_{\vec{x}} = 1$$

$$\Rightarrow 2\pi\varepsilon \left.\frac{\partial G}{\partial r}\right|_{r=\varepsilon} = 1 \Rightarrow \left.\frac{\partial G}{\partial r}\right|_{r=\varepsilon} = \frac{1}{2\pi\varepsilon}$$

From (10.1) $\left.\frac{\partial G}{\partial r}\right|_{r=\varepsilon} = \frac{C_1}{\varepsilon} \Rightarrow C_1 = \frac{1}{2\pi}.$

$$\Rightarrow \quad G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln(r) + C_2 .$$

unit except a constant C_2. For convenience, $\underline{C_2 = 0}$.

$$\Rightarrow \quad \boxed{G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln(r) \quad \text{in } \mathbb{R}^2.}$$

Now, using Green's formula over a large circle of radius r.

$$\iint_{\Omega} [u\nabla^2 G - G\nabla^2 u]\, dA_{\vec{x}} = \oint_{C_r} (u\nabla G - G\nabla u)\cdot\hat{n}\, dl_{\vec{x}}$$

We will impose the condition that $\downarrow$ 0. when $r \to \infty$.

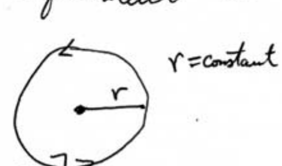
$$\Rightarrow \quad \boxed{u(\vec{x}) = \iint_{\mathbb{R}^2} G(\vec{x},\vec{x}_0)\, f(\vec{x}_0)\, dS_{\vec{x}_0}}$$

(reversing roles $x \Leftrightarrow x_0$ and using symmetry $G(\vec{x},\vec{x}_0) = G(\vec{x}_0,\vec{x})$)

<u>Rmk</u>: The additional condition

$$\lim_{r\to\infty} \oint_{C_r} (u\nabla G - G\nabla u)\cdot\hat{n}\, dl_{\vec{x}} = 0 \quad (\hat{n} = \hat{r})$$

is equivalent to



$$\lim_{r\to\infty} \int_{r\ \text{constant}} u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\, \underbrace{dl_{\vec{x}}}_{r\,d\theta} = 2\pi r\left[u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right] \xrightarrow[r\to\infty]{} 0$$

Or simply

$$\lim_{r\to\infty} r\left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right) = \lim_{r\to\infty} r\left[u\frac{1}{\cancel{2\pi} r} - \frac{1}{\cancel{2\pi}}\ln(r)\frac{\partial u}{\partial r}\right] = 0$$

$$\Rightarrow \lim_{r\to\infty}\left[u - r\ln(r)\frac{\partial u}{\partial r}\right] = 0.$$

This condition is satisfied for example if $u \sim \frac{1}{r}$ when $r\to\infty$.