Green's function for bounded domain using infinite Space Green's function.

$$\begin{cases} \nabla^2 G = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \Omega \text{ (bounded domain)} \quad (1) \\ + \text{homog BC's.} & (2) \end{cases}$$

Idea is to use infinite space Green's fn. + Something else

$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_0| + v(\vec{x},\vec{x}_0).$$

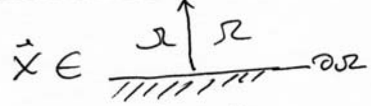
Where $v(\vec{x},\vec{x}_0)$ Satisfies

$$\begin{cases} \nabla^2 v = 0, & \vec{x} \in \Omega \quad (3) \\ v(\vec{x}_s,\vec{x}_0) = -\frac{1}{2\pi} \ln|\vec{x}_s-\vec{x}_0|, & \vec{x}_s \in \partial\Omega \quad (4) \end{cases}$$

If boundary is Simple (3) and (4) Can be Solved by Separation of variables. In general, it's not easy to Solve (3) and (4).

Method of Images. Semi-infinite plane $(y>0)$.

Consider
$$\begin{cases} \nabla^2 u = f(x), & \vec{x} \in \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega = \{(x,y) \mid y=0\} \end{cases}$$

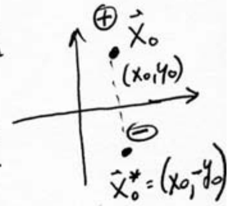


Corresponding Green's function satisfies:

$$\begin{cases} \nabla^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), & \vec{x} \in \Omega = \\ G(x, 0; x_0, y_0) = 0 , & \end{cases}$$

In this case, there is an easy way to construct $v(\vec{x})$ satisfying (3) and (4).

Consider the infinite space Green's function



$$\nabla^2 \bar{G}(\vec{x}, \vec{x}_0^*) = -\delta(\vec{x} - \vec{x}_0^*), \qquad \vec{x} \in \mathbb{R}^2.$$

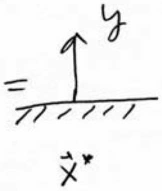
where $\vec{x}_0^* = (x_0, -y_0)$. $\Rightarrow$ $x_0^* = x_0$, $y_0^* = -y_0$

Then, $$\boxed{\bar{G}(\vec{x}, \vec{x}_0^*) = -\frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0^*|} \qquad (5)$$

and at the x-axis $= \{ (x, y)$ such that $\underline{y = 0} \}$

$$\boxed{\bar{G}(x, 0; x_0^*, y_0^*) = -\frac{1}{2\pi} \ln \left[(x - x_0)^2 + y_0^2 \right]^{1/2}} \qquad (6)$$

Also, $\nabla^2 \bar{G}(\vec{x}, \vec{x}_0^*) = 0$, $\quad \vec{x} \in \Omega =$



because $\vec{x}^*$ is in the lower half plane. $\vec{x}^* \notin$ Upper half plane $\Rightarrow$ $\delta(\vec{x} - \vec{x}_0^*) = 0$ for $\vec{x} \in \Omega$.

Therefore, the restriction of $\bar{G}(\vec{x},\vec{x}_0^*)$ to the half plane $(y>0)$ is the function $v(\vec{x})$ sought, which satisfies (3) and (4) $\quad \vec{x}_0^* = (x_0, -y_0)$

$$\begin{cases} \nabla^2 \bar{G}(\vec{x},\vec{x}_0^*) = 0, & \{(x,y) \mid y>0\} = \Omega \\ \bar{G}(x,0;\underset{x_0}{\underset{\shortparallel}{x_0^*}},\underset{-y_0}{\underset{\shortparallel}{y_0^*}}) = -\frac{1}{2\pi}\ln\left(\sqrt{(x-x_0)^2+y_0^2}\right) & \end{cases}$$

Therefore, the Green's function for the half plane is given by

$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0^*| =$$

$$= \frac{1}{2\pi}\ln\left[\frac{\left((x-x_0)^2+(y-y_0)^2\right)^{1/2}}{\left((x-x_0)^2+(y+y_0)^2\right)^{1/2}}\right]$$

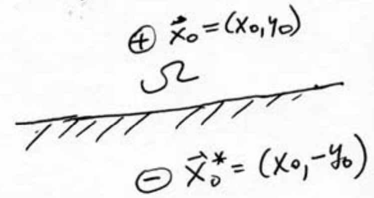
$$\therefore \quad \boxed{G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right]} \qquad (7)$$

Verification:

$$\nabla^2 G(\vec{x},\vec{x}_0) = \nabla^2\left(\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0|\right) - \nabla^2\left(\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0^*|\right)$$

$$= \delta(\vec{x}-\vec{x}_0) - \delta(\vec{x}-\vec{x}_0^*) = \delta(\vec{x}-\vec{x}_0)$$

for $\vec{x} \in \{\text{Half plane, } y>0\}$.



At the boundary $\vec{x}_s=(x,0)$.

Also, $G(\vec{x}_s,\vec{x}_0) = G(x,0;x_0,y_0) =$

$$= \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+y_0^2}{(x-x_0)^2+y_0^2}\right] = \frac{1}{4\pi}\ln(1) = 0. \quad \checkmark$$

Back to Nonhomogeneous problem:

$$\begin{cases} \nabla^2 u = f(x), & + \text{ B.C. } u(x,0)=h(x). \\ & \Omega \equiv \text{Half plane } y>0. \end{cases}$$

Use Green's formula:

$$\iint_{\text{Half plane}} \left[u\nabla^2 G - G\nabla^2 u\right] ds_{\vec{x}} = \oint_{y=0,\ r\to\infty} \left[u\nabla G - G\nabla u\right]\cdot\hat{n}\, dl_{\vec{x}}$$

or
$$\iint \left[u\, \delta(\vec{x}-\vec{x}_0) - G\, f(\vec{x}) \right] ds_{\vec{x}} = \int_{-\infty}^{\infty} \left[u\left(-\frac{\partial G}{\partial y}\right) - \overset{=0}{G}\left(-\frac{\partial u}{\partial y}\right) \right] dx$$
$$+ \int_0^{\pi} \left[u \frac{\partial G}{\partial r} - G \frac{\partial u}{\partial r} \right] r \Big|_{r \to \infty} d\theta \;\to 0.$$

Def of infinite space Green's fn.

$$\Rightarrow \quad u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0)\, f(\vec{x})\, ds_{\vec{x}} - \int_{-\infty}^{\infty} h(\vec{x}) \frac{\partial G}{\partial y}(\vec{x},\vec{x}_0)\Big|_{y=0} dx$$

reversing roles $x \to x_0$ + Symmetry of G.

$$u(\vec{x}) = \iint G(\vec{x},\vec{x}_0)\, f(\vec{x}_0)\, ds_{\vec{x}_0} - \int_{-\infty}^{\infty} h(\vec{x}_0) \frac{\partial G}{\partial y_0}(\vec{x},\vec{x}_0)\Big|_{y_0=0} dx_0$$

Now, $\displaystyle \frac{\partial G}{\partial y}(\vec{x},\vec{x}_0)\Big|_{y_0=0} = \frac{\partial G}{\partial y}(x,y;x_0,0) = \frac{-y/\pi}{(x-x_0)^2 + y^2}$

In fact,

$$G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\ln\left[(x-x_0)^2+(y-y_0)^2\right] - \ln\left[(x-x_0)^2+(y+y_0)^2\right]\right]$$

$$\Rightarrow \quad \frac{\partial}{\partial y_0} G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}\right]$$

$$\Rightarrow \quad \frac{\partial G}{\partial y_0}(x,y;x_0,0) = \underbrace{\frac{1}{\cancel{4}\pi}\left[\frac{-\cancel{4}y}{(x-x_0)^2+y^2}\right]}_{\text{dipole source.}}$$

In particular,
the BVP in the half plane given by

$$\begin{cases} \nabla^2 u = 0 \\ u(x,0) = h(x) \end{cases}$$

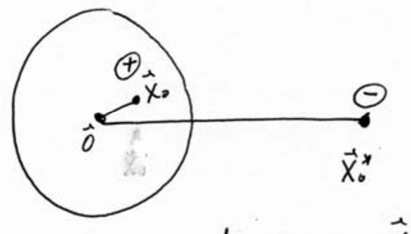
has as a solution:

$$u(x,y) = -\int_{-\infty}^{\infty} \left(h(\vec{x}_0)\ \frac{\partial G}{\partial y_0}(\vec{x},\vec{x}_0)\right)\Big|\ dx_0 =$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty} h(x_0)\ \frac{1}{(x-x_0)^2+y^2}\ dx_0$$

Method of Images. Green's function for a Circle.

$$\begin{cases} \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \text{Circle of radius } a. \quad (1) \\ G(\vec{x},\vec{x}_0) = 0, & |\vec{x}| = a \quad (2) \end{cases}$$



The idea is to find $\vec{x}_0^*$ somewhere outside the circle and define $\bar{G}(\vec{x},\vec{x}_0)$ in the whole plane $\mathbb{R}^2$ satisfying

$$\begin{cases} \nabla^2 \bar{G}(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) - \delta(\vec{x}-\vec{x}_0^*), & \vec{x} \in \mathbb{R}^2. \quad (3) \\ \bar{G}(\vec{x}_s,\vec{x}_0) = 0, & |\vec{x}_s| = a. \quad (4) \end{cases}$$

This $\bar{G}(\vec{x},\vec{x}_0)$ restricted to the region bounded by the circumference $r=a$ is the Green's function for the circle with homog. Dirichlet B.C., i.e.,

$$G(\vec{x},\vec{x}_0) = \bar{G}(\vec{x},\vec{x}_0)\Big|_{\Omega_a}$$

The solution of (3) and (4) can be obtained as a superposition of a positive source located at $\vec{x}_0$ and of a negative source located at $\vec{x}_0^*$.

$$\boxed{G(\vec{x},\vec{x}_0) = \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_0^*| + C} \quad (5)$$

where C is a constant to be determined.

Then, we have two unknowns: $\vec{x}_0^*$ and C.

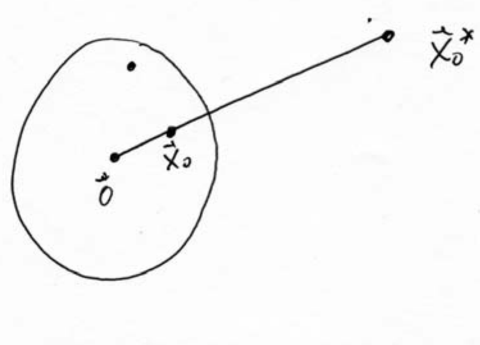
We also have one condition:

$$\boxed{G(\vec{x},\vec{x}_0)\Big|_{|\vec{x}|=a} = 0} \quad (6)$$

And an additional condition is that

$$\boxed{\vec{x}_0^* = \gamma \vec{x}_0} \quad (7)$$

It means that the image point is along the same radial line as the source point $\vec{x}_0$.



Condition (6) $G\big|_{r=a} = 0$ is satisfied by (5) if

$$G(\vec{x},\vec{x}_0) = \frac{1}{4\pi} \ln \frac{|\vec{x}-\vec{x}_0|^2}{|\vec{x}-\vec{x}_0^*|^2}\Bigg|_{|\vec{x}|=a} + C = 0$$

$$\Rightarrow \quad C = -\frac{1}{4\pi} \ln \frac{|\vec{x}-\vec{x}_0|^2}{|\vec{x}-\vec{x}_0^*|^2}\Bigg|_{|\vec{x}|=a} \Rightarrow -4\pi C = \ln\left(\frac{\quad}{\quad}\right)$$

$$\Rightarrow \quad e^{-4\pi C} = \frac{|\vec{x}-x_0|^2}{|\vec{x}-\vec{x}_0^*|^2}\Bigg|_{|\vec{x}|=a}$$

$$\Rightarrow \quad |\vec{x}-\vec{x}_0|^2\Big|_{|\vec{x}|=a} = \underbrace{e^{-4\pi C}}_{\substack{\parallel \\ K}} |\vec{x}-\vec{x}_0^*|^2\Big|_{|\vec{x}|=a}$$

$$\boxed{|\vec{x}-\vec{x}_0|^2 = K\,|\vec{x}-\vec{x}_0^*|^2} \quad \text{when } |\vec{x}|=a \qquad (8).$$

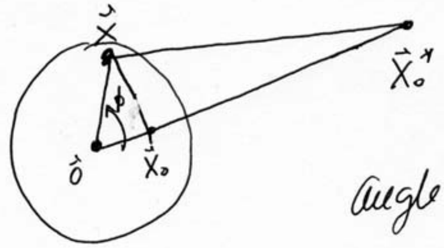
Now,

$$|\vec{x}-\vec{x}_0|^2 = (\vec{x}-\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) = |\vec{x}|^2 + |\vec{x}_0|^2 - 2\,\vec{x}\cdot\vec{x}_0.$$

$$= |\vec{x}|^2 + |\vec{x}_0|^2 - 2\,|\vec{x}|\,|\vec{x}_0| \cos\phi.$$

Similarly,

$$|\vec{x}-\vec{x}_0^*|^2 = |\vec{x}|^2 + |\vec{x}_0^*|^2 - 2\,|\vec{x}|\,|\vec{x}_0^*| \cos\phi.$$



angle betw $\vec{x}$ and $\vec{x}_0$

Same as " betw $\vec{x}$ and $\vec{x}_0^*$

Over the circle $|\vec{x}| = a$

$$|\vec{x}_0^*| = \gamma |\vec{x}_0| = \gamma r_0, \text{ where } |\vec{x}_0| = r_0$$

and equ. (8) holds. Therefore,

$$a^2 + r_0^2 - 2ar_0\cos\phi = K\left(a^2 + \gamma^2 r_0^2 - 2a\gamma r_0 \cos\phi\right)$$

true for all ϕ.

$$\Rightarrow \begin{cases} a^2 + r_0^2 = K\left(a^2 + \gamma^2 r_0^2\right) & (*.1) \quad \left(\phi = \frac{\pi}{2}\right) \\ \text{and} \quad 2ar_0 = \left(2a\gamma r_0\right)K & (*.2) \end{cases}$$

From $(*.2)$

$$\Rightarrow \boxed{K = 1/\gamma}$$

Subst. in $(*.1)$: $a_0^2 + r_0^2 = \frac{1}{\gamma}a^2 + \gamma r_0^2$

$$\Rightarrow \boxed{\gamma = a^2/r_0^2}$$

$$\Rightarrow \boxed{\vec{x}_0^* = \frac{a^2}{r_0^2}\vec{x}_0}$$

Location of negative point source outside circle.

$$\Rightarrow \quad k = \frac{r_0^2}{a^2} \quad \Rightarrow \quad \begin{array}{l} k = e^{-4\pi c} \\ \dfrac{r_0^2}{a^2} = e^{-4\pi c} \Rightarrow \end{array}$$

$$\Rightarrow \quad -4\pi c = \ln\left(\frac{r_0^2}{a^2}\right) \Rightarrow c = -\frac{1}{4\pi}\ln\left(\frac{r_0^2}{a^2}\right) =$$

$$\therefore \quad \boxed{c = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\right)}$$

Therefore, Subst. in (5) we can obtain the Green's fn. for our problem (Circle of radius a DIRICHLET COND.)

$$\boxed{G(x, \vec{x}_0) = \frac{1}{4\pi}\ln\frac{|\vec{x}-x_0|^2}{|\vec{x}-x_0^*|^2} + \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\right)}$$

where $\vec{x}_0^* = \dfrac{a^2}{r_0^2}\vec{x}_0$. $\Rightarrow$ $r_0^* = \dfrac{a^2}{r_0^2} r_0 = \dfrac{a^2}{r_0}$

or $$\boxed{G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi}\ln\left(\frac{|\vec{x}-\vec{x}_0|^2\, a^2}{|\vec{x}-\vec{x}_0^*|^2\, r_0^2}\right)}$$

or $$G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\;\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2+r_0^{*2}-2rr_0^*\cos\phi}\right)$$

$$= \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\;\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2+\dfrac{a^4}{r_0^2}-2r\dfrac{a^2}{r_0}\cos\phi}\right)$$

$$\therefore \boxed{G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\ln\left(a^2\,\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2r_0^2+a^4-2rr_0a^2\cos\phi}\right)} \quad (9)$$

Then, the solution for the nonhomogeneous BVP.

$$\begin{cases} \nabla^2 u = f(\vec{x}), & \vec{x} \in \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega = \end{cases}$$

is given by $\boxed{u(\vec{x}) = \iint f(\vec{x}_0)\,G(\vec{x},\vec{x}_0)\,ds_{\vec{x}_0} + \oint h(\vec{x}_0)\,\nabla_{\vec{x}_0}G(\vec{x},\vec{x}_0)\cdot\hat{n}\,dl_{\vec{x}_0}}$

or $$\boxed{u(\vec{x}) = \int_0^a\int_0^{2\pi} f(\vec{x}_0)\,G(\vec{x},\vec{x}_0)\,r\,d\theta\,dr + \int_0^{2\pi} h(\vec{x}_0)\,\frac{\partial G}{\partial r_0}(\vec{x},\vec{x}_0)\Big|_{r_0=a}\,a\,d\theta} \quad (10)$$

Now,

$$\frac{\partial G}{\partial r_0} = \frac{1}{4\pi}\left(\frac{2r_0-2r\cos\phi}{r^2+r_0^2-2rr_0\cos\phi} - \frac{2r^2r_0-2ra^2\cos\phi}{r^2r_0^2+a^4-2rr_0a^2\cos\phi}\right)$$

$$r^2r_0^2+a^4-2rr_0a^2\cos\phi = r_0^2\left(r^2+a^4/r_0^2-2r\frac{a^2}{r_0}\cos\phi\right)$$

$$\Rightarrow \frac{\partial G}{\partial r_0}(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left(\frac{2r_0-2r\cos\phi}{r^2+r_0^2-2rr_0\cos\phi} - \frac{2r^2/r_0-2r\frac{a^2}{r_0^2}\cos\phi}{r^2+\frac{a^4}{r_0^2}-2r\frac{a^2}{r_0}\cos\phi}\right)$$

$$\Rightarrow \frac{\partial G}{\partial r_0}\Big|_{r_0=a} = \frac{1}{4\pi}\left(\frac{2a-\cancel{2r\cos\phi}-2r^2/a+\cancel{2r\cos\phi}}{r^2+a^2-2ra\cos\phi}\right)$$

$$\therefore \quad \left.\frac{\partial G}{\partial r_0}\right|_{r_0=a} = \frac{2a}{4\pi}\left(\frac{1-(r/a)^2}{r^2+a^2-2ra\cos\phi}\right)$$

Therefore, if our problem is

$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases}$$

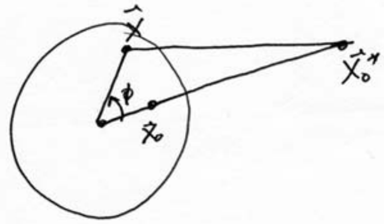
then from (10).

$$u(\vec{x}) = u(r,\theta) = \frac{\cancel{a}}{2\pi}\int_0^{2\pi} h(\theta_0)\,\frac{(a^2-r^2)/\cancel{a^2}}{r^2+a^2-2ra\cos\phi}\,\cancel{a}\,d\theta_0$$

or

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta_0)\,\frac{a^2-r^2}{r^2+a^2-2ra\cos\phi}\,d\theta_0}$$

Now,



$\phi = \theta - \theta_0$

To obtain (9.5.61) in book.

Hwk problems due Next Wed: 10 Haberman 9.5.5, 9.5.7, 9.5.10, 9.5.11, 9.5.14, 9.5.16.