Generalized Green's Functions.

Consider the nonhomogeneous problem

$$\begin{cases} Au = f(x), & a < x < b \quad (1) \\ B_1 u(a) = 0, \quad B_2 u(b) = 0 & \quad (2) \end{cases}$$

when $\lambda = 0$ is an eigenvalue.

We want to find out the solution in terms of a "generalized Green's function".

We have seen that (1) and (2) has no solutions or infinitely many solutions. Let's assume that $\phi_h(x)$ is an eigenfunction corresponding to $\lambda = 0$, or equivalently that $\phi_h(x)$ is a nontrivial solution of the corresponding homogeneous problem.

To have infinitely many solutions

$$\int_a^b f(x)\, \phi_h(x)\, dx = 0 \qquad u = \sum_1^{\infty} u_n \phi_n$$

Recall that $u_n \lambda_n = f_n$

If $\lambda_n = 0 \Rightarrow f_n = 0$ for solutions to exist.

We will illustrate the process of construction of a Generalized Green's function and then construction of solutions of (1) and (2) through the following example:

$$\begin{cases} \dfrac{d^2u}{dx^2} = f(x) = x - \dfrac{L}{2}, & 0 < x < L. \\ \dfrac{du}{dx}(0) = 0, \quad \dfrac{du}{dx}(L) = 0 \end{cases}$$

Obviously, there are nontrivial solutions for the corresponding homogeneous problem : $\phi_h(x) \equiv 1$. and any multiple

Therefore, for a solution to exist

$$\boxed{\int_0^L f(x)\,\phi_h(x)\,dx = \int_0^L f(x)\,dx = 0}$$

If $f(x) = x - L/2 \Rightarrow \int_0^L \left(x - \frac{L}{2}\right)dx = \left.\frac{x^2}{2} - \frac{L}{2}x\right|_0^L = 0$ ✓

We observe that the Green's fn. for the BVP given does not exist. In fact, for

(for any x_0) $\quad \dfrac{d^2G}{dx^2} = \delta(x - x_0), \quad u_G'(0) = 0, \quad u_G'(L) = 0 \quad (*)$

$$\int_0^L \delta(x - x_0)\,\underbrace{\phi_h(x)}_{\equiv 1}\,dx = \phi_h(x_0) \equiv 1 \neq 0$$

So $\delta(x - x_0) \not\perp \phi_h(x) \equiv 1 \Rightarrow$ There is no soln. (Green's fn.) for $(*)$.

However, we can redefine rhs of (*) to achieve orthogonality as follows:

$$\frac{d^2 G_m}{dx^2} = \delta(x-x_0) + \overset{?}{C}\,\phi_h(x)$$

C is determined from the orthogonality condition

$$0 = \int_0^L \phi_h(x)\left[\delta(x-x_0) + C\,\phi_h(x)\right]dx =$$

$$\Rightarrow \quad 0 = \phi_h(x_0) + C\int_0^L \phi_h^2(x)\,dx$$

$$\Rightarrow \quad C = \frac{-\phi_h(x_0)}{\int_0^L \phi_h^2(x)\,dx \neq 0}, \text{ because } \phi_h(x) \not\equiv 0 \text{ nontrivial soln.}$$

and the related BVP. (nonhomog.)

$$\begin{cases} \dfrac{d^2 G_m}{dx^2} = \underbrace{\delta(x-x_0) - \left(\phi_h(x_0)\Big/\int_0^L \phi_h^2(x)\,dx\right)\phi_h(x)}_{\downarrow} \\ G_m'(0) = 0, \quad G_m'(L) = 0 \end{cases}$$

has nontrivial solutions, because rhs $\perp$ $\phi_h(x)$.

In our case, $\phi_h(x) \equiv 1$

$$\Rightarrow \quad C = \frac{-1}{L}$$

and related BVP.

$$\begin{cases} \dfrac{d^2 G_m}{dx^2} = \delta(x-x_0) - \dfrac{1}{L} \\ G_m'(0) = 0, \quad G_m'(L) = 0 \end{cases}$$

If $x \neq x_0$ $\quad \dfrac{d^2 G_m}{dx^2} = -\dfrac{1}{L}$

by integrating $\quad \dfrac{dG_m(x)}{dx} = -\dfrac{1}{L}x + d_{l,r}$

To satisfy BC at the left.

$$0 = G_m'(0) = d_l \Rightarrow \boxed{d_l = 0}$$

To satisfy BC. at the right

$$0 = G_m'(L) = -\frac{L}{L} + d_r \Rightarrow \boxed{d_r = 1}$$

$$\Rightarrow \quad G_m'(x) = \left\{ \begin{array}{ll} -\dfrac{x}{L}, & x < x_0 \\ -\dfrac{x}{L} + 1, & x > x_0 \end{array} \right\} \quad (**)$$

The jump condition in the derivative at $x = x_0$ $\quad \int_{x_0-\varepsilon}^{x_0+\varepsilon}$

$$\frac{dG_m}{dx}(x_0+\varepsilon) - \frac{dG_m}{dx}(x_0-\varepsilon) = -\frac{1}{L}\left[\cancel{x_0}+\varepsilon - \cancel{x_0}+\varepsilon\right] + 1$$

$$\underset{\varepsilon \to 0}{\Rightarrow} \boxed{G_m'(x_0^+) - G_m'(x_0^-) = 1}$$

It's automatically satisfied.

Integrating again $(**)$ $\int dx$

$$G_m(x, x_0) = \begin{cases} -\frac{x^2}{2}\frac{1}{L} + h_1(x_0) & x < x_0 \\ -\frac{x^2}{2}\frac{1}{L} + x + h_2(x_0) & x > x_0 \end{cases}$$

Cont. at x_0 $\Rightarrow$ $G_m(x_0^+, x_0) = G_m(x_0^-, x_0)$.

$$\Rightarrow \quad \cancel{-\frac{x_0^{+2}}{2}\frac{1}{L}} + x_0^+ + h_2(x_0) = \cancel{-\frac{x_0^{-2}}{2}\frac{1}{L}} + h_1(x_0)$$

$$\Rightarrow \boxed{h_1(x_0) - h_2(x_0) = x_0} \Rightarrow h_1(x_0) = x_0 + h_2(x_0)$$

or

$$G_m(x, x_0) = \begin{cases} -\frac{x^2}{2}\frac{1}{L} + x_0 + h_2(x_0), & x < x_0 \\ -\frac{x^2}{2}\frac{1}{L} + x + h_2(x_0), & x > x_0 \end{cases}$$

$h_2 \to h_2(x_0)$

by requiring symmetry:

$$\underbrace{-\frac{x^2}{2}\frac{1}{L} + \cancel{x_0} + h_2(x_0)}_{G_m(x, x_0)} = \underbrace{-\frac{x_0}{2}\frac{1}{L} + \cancel{x_0} + h_2(x)}_{G_m(x_0, x)}, \quad x < x_0$$

$$\Rightarrow h_2(x_0) = -\frac{1}{L}\frac{x_0^2}{2} + \beta$$

$h_2(x_0) + \frac{x_0}{2}\frac{1}{L} = = h_2(x) + \frac{x}{2}\frac{1}{L} = \beta$

$$\Rightarrow \quad G_m(x, x_0) = \begin{cases} -\frac{1}{L}\frac{x^2 + x_0^2}{2} + x_0 + \beta, & x < x_0 \\ -\frac{1}{L}\frac{x^2 + x_0^2}{2} + x + \beta, & x > x_0 \end{cases}$$

Now, a soln. (among infinitely many) for the BVP given is

$$u(x) = \int_0^L f(x_0)\, G_m(x, x_0)\, dx_0$$

In fact, using Green's formula:

$$\int_a^b [u L v - v L u]\, dx = \left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right]_a^b$$

homog BC's

$$\int_a^b \Big\{ u(x) \underbrace{\left[\delta(x - x_0) - \frac{\phi_h(x)\, \phi_h(x_0)}{\int_a^b \phi_h^2(x_0)\, dx_0} \right]}_{L[G_m(x,x_0)]} - G_m(x, x_0) \underbrace{f(x)}_{Lu} \Big\}\, dx = 0.$$

$$\Rightarrow \quad u(x_0) = \frac{\phi_h(x_0)}{\int_a^b \phi_h^2(x_0)\, dx_0} \int_a^b \phi_h(x)\, u(x)\, dx + \int_a^b G_m(x, x_0)\, f(x)\, dx$$

Renaming Variables $x_0 \leftrightarrow x$. and using symmetry of $G_m(x, x_0)$

$$u(x) = \underbrace{\frac{\phi_h(x)}{\int_a^b \phi_h^2(\bar{x})\, d\bar{x}} \int_a^b \phi_h(x_0)\, u(x_0)\, dx_0}_{\text{multiple of homog. soln.}} + \int_a^b G_m(x, x_0)\, f(x_0)\, dx_0$$

So particular soln.

$$\boxed{u(x) = \int_a^b G_m(x, x_0)\, f(x_0)\, dx_0}$$