Method of Separation of variables applied to the BVP (14).

$$\psi(r,\theta) = \phi(\theta)\, G(r) \qquad (15.1)$$

$$\nabla^2_{r,\theta}\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad (15.2)$$

Substituting (15.1) into (15.2)

$$\phi(\theta)\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$$

multiplying by $\dfrac{r^2}{\phi(\theta)G(r)}$

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{1}{\phi(\theta)}\frac{d^2\phi}{d\theta^2} = 0$$

or $$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi(\theta)}\frac{d^2\phi}{d\theta^2} = \lambda$$

Thus,
$$\frac{d^2\phi}{d\theta^2} + \lambda\phi = 0 \qquad (16.1)$$

and
$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0 \qquad (16.2)$$

$\phi(\theta)$ also satisfies periodic B.c's.

$$\phi(-\pi) = \phi(\pi) \quad \text{and} \quad \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

What is the meaning of these conditions?

Therefore, $\phi(\theta)$ satisfies the eigenvalue problem:

$$\begin{cases} \dfrac{d^2\phi}{d\theta^2} + \lambda\phi = 0 , & -\pi < \theta < \pi. \\ \phi(-\pi) = \phi(\pi) , & \phi'(-\pi) = \phi'(\pi) \end{cases}$$

This is a well known eigenvalue problem whose eigenvalues and eigenfunctions are (See Haberman)

$$\lambda_n = n^2, \quad n = 0, 1, 2, \dots$$

and
$$\phi_n(\theta) = \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}_{n=0}^{\infty}$$

The equations for $G(r)$ are now given by

$$r \frac{d}{dr}\left(r \frac{dG}{dr}\right) - n^2 G = 0 , \quad n = 0, 1, 2, \ldots \qquad (17.1)$$

If $\underline{n=0}$ $\quad r \frac{d}{dr}\left(r \frac{dG}{dr}\right) = 0$

$$\therefore \quad \frac{d}{dr}\left(r \frac{dG}{dr}\right) = 0 \Rightarrow r \frac{dG}{dr} = C_1$$

$$\Rightarrow \quad \frac{dG}{dr} = \frac{C_1}{r} \Rightarrow \boxed{G(r) = C_1 \ln(r) + C_2} \qquad (17.2)$$

If $\underline{n \neq 0}$

Equ. (17.1) may be written as

$$r^2 \frac{d^2G}{dr^2} + r \frac{dG}{dr} - n^2 G = 0$$

which is Euler equation. Look for solutions in the form: $G(r) = r^p$, and find out $p = \pm n$

Since $G_1(r) = r^n$ and $G_2(r) = r^{-n}$ are linearly independent solutions of (17.1) for $n \neq 0$, the general solution can be written as

$$\boxed{G(r) = C_1 r^n + C_2 r^{-n}} \qquad (17.3)$$

Applying the principle of Superposition, we obtain as solution of Laplace's equation for $\psi(r,\theta)$

$$\psi(r,\theta) = C_1 \ln r + C_2 + \sum_{n=1}^{\infty} \left[A_n r^n + B_n r^{-n}\right] \sin(n\theta) + $$
$$+ \sum_{n=1}^{\infty} \left[D_n r^n + E_n r^{-n}\right] \cos(n\theta). \qquad (18.1)$$

<u>Determining constants from B.C's:</u>

B.C. (14.3): $\psi(r,\theta) \xrightarrow[r\to\infty]{} y = r\sin\theta$

It's equivalent to

$$C_1 \ln r + C_2 + A_1 r \sin\theta + B_1 \frac{1}{r}\sin\theta + A_2 r^2 \sin(2\theta) + B_2 \frac{1}{r^2}\sin(2\theta) + \cdots$$
$$+ D_1 r\cos\theta + E_1 \frac{1}{r}\cos\theta + D_2 r^2 \cos(2\theta) + E_2 \frac{1}{r^2}\cos(2\theta) + \cdots$$
$$\Big\downarrow r\to\infty$$
$$r\sin\theta$$

Therefore,

$A_1 = 1$,

B_n $(n \geq 1)$; they are all undetermined, since $B_n \frac{1}{r^n} \sin(n\theta) \xrightarrow[r\to\infty]{} 0$

$D_1 = 0$; because the dominant and only term in the right hand side is $r\sin\theta$.
The term $D_1 r\cos\theta$ is of the same $\mathcal{O}(r)$, but it would not have a counterpart in the rhs.

$D_n = 0$ $(n \geq 2)$; because they are $\mathcal{O}(r^n)$ and they would dominate the term $r\sin\theta$ in the rhs.

E_n $(n \geq 1)$: They are undetermined because all terms $E_n \frac{1}{r^n} \cos(n\theta) \xrightarrow[r\to\infty]{} 0$

Finally, the terms $C_1 \ln r + C_2$ are dominated by $r\sin\theta$. More precisely

$$C_1 \ln r + C_2 + r\sin\theta \xrightarrow[r\to\infty]{} r\sin\theta.$$

Therefore, C_1 and C_2 remain undetermined.

Thus, the Solution (18.1) reduces to

$$\psi(r,\theta) = C_1 \ln r + C_2 + \left(r + B_1 \frac{1}{r}\right)\sin\theta + $$

$$+ \sum_{n=2}^{\infty} B_n r^{-n} \sin(n\theta) + \sum_{n=1}^{\infty} E_n r^{-n} \cos(n\theta).$$

Applying B.C. (14.2): $\psi(1,\theta) = 0$

we obtain

$$C_1 \overset{0}{\cancel{\ln(1)}} + C_2 + (1+B_1)\sin\theta + B_2 \sin(2\theta) + \cdots$$

$$+ E_1 \cos\theta + E_2 \cos(2\theta) + \cdots = 0$$

Then $C_2 = 0$, $B_1 = -1$, $B_n = 0$, $n \geq 2$
$E_n = 0$, $n \geq 1$

The final answer is given by

$$\boxed{\psi(r,\theta) = C_1 \ln r + \left(r - \frac{1}{r}\right)\sin\theta} \qquad (20.1)$$

Therefore, there are infinitely many solutions depending on C_1.