Math 521
METHODS OF APPLIED MATHEMATICS
Homework 1: Review of Vector Calculus

1. Using a Cartesian coordinate system, verify the following vector identities:

   (a) $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$ (A.15)

   (b) $(\mathbf{A}.\nabla)\,\mathbf{A} = \nabla \left(\frac{|\mathbf{A}|^2}{2}\right) - \mathbf{A} \times (\nabla \times \mathbf{A})$ (A.16)

   (c) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}.\nabla)\,\mathbf{A} - \mathbf{B}\,(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla \cdot \mathbf{B})$ (A.17a)

2. (a) Use (A.15) to prove that

   $$\nabla \times \left(\nabla^2 \mathbf{u}\right) = \nabla^2 (\nabla \times \mathbf{u})\,.$$

   (b) Use some of the previous identities to show that

   $$\nabla \times ((\mathbf{u} \cdot \nabla)\,\mathbf{u}) = -\,(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} + (\mathbf{u} \cdot \nabla)\,\boldsymbol{\omega},$$

   where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

   (c) Using (a) and (b) obtain (10.5) from (10.1) and (10.2) (in Leal's book)

3. If flow is two dimensional

   (a) Show that $\boldsymbol{\omega} = \omega \mathbf{k}$
   It means the vorticity only has a nonzero component and is normal to the plane of flow.

   (b) Show that $(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} = 0$.

   (c) Knowing that $\nabla \times \mathbf{u} = \boldsymbol{\omega}(x, y) = 0$, everywhere in the unbounded region $x^2 + y^2 > 1$ and $\mathbf{u}(x, y) = \nabla \times (\psi(x, y)\mathbf{k})$. Show (either directly or using one of the previous vector identities) that

   $$\nabla^2 \psi(x, y) = 0, \qquad \text{everywhere on } x^2 + y^2 > 1.$$

4. Use the method of separation of variable to obtain the solution of the exterior problem for the stream function, ψ.

   $$\begin{cases} \nabla^2_{r,\theta}\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 & (2.1) \\ \psi(1, \theta) = 0 \qquad\qquad x^2 + y^2 > 1 & (2.2) \\ \psi(r, \theta) \to y = r \sin\theta, \qquad \text{when } r \to \infty & (2.3) \end{cases}$$

   (a) Introduce $\psi(r, \theta) = \Phi(\theta) G(r)$
   and obtain an eigenvalue problem for $\Phi(\theta)$, and a second order equation for $G(r)$.

(b) Solve the eigenvalue problem for $\Phi(\theta)$ and obtain a general solution for $G(r)$ for each eigenvalue.

(c) Apply the principle of superposition to obtain a solution for (2.1). <u>At this stage keep all unknown constants.</u>

(d) Determine all the unknowns constants from B.C's (2.2) and (2.3).

5. Using suffix notation prove the following vector identities:

   (a) (Problem 1. (c) above)
   $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$

   (b) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

   (c) $[\mathbf{u} \times (\nabla \times \mathbf{v})]_i = \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x_i} - [(\mathbf{u} \cdot \nabla)\mathbf{v}]_i$
   Use this identity (c) to obtain
   $\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}$