Suffix notation in Vector Calculus.

$$\vec{C} = \vec{a} + \vec{b} \longrightarrow C_i = a_i + b_i$$

$i = 1, 2, 3.$

Dot product:

$$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \Longrightarrow a_i b_i$$

Summation Convention: Any time a suffix is repeated then the term is summed from $i=1$ to $i=3$.

Ask for $(\vec{a}.\vec{b})(\vec{c}.\vec{d}) = a_i b_i c_j d_j$

Example: Write vector equation

$$\vec{u} + (\vec{a}.\vec{b})\vec{v} = |a|^2 (\vec{b}.\vec{v})\vec{a}$$

$$\downarrow$$

$$u_i + a_j b_j v_i = a_j a_j b_k v_k a_i$$

Delta Kronecker

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \delta_{ij} = \delta_{ji}$$

$$\delta_{ij}\, a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 = \underline{\underline{a_i}} \; ?$$

"Substitution tensor"

$$\delta_{ij}\, a_j = a_i$$

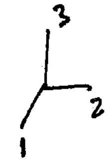
$$\delta_{ij}\, a_i = a_j$$

$$\vec{a} \cdot \vec{b} = \delta_{ij}\, a_j\, b_i = \delta_{ij}\, a_i\, b_j$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Tensor ϵ_{ijk}

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any of } i, j, k \text{ are equal} \\ 1, & \text{if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1, & \text{if } (i,j,k) = (3,2,1), (2,1,3), (1,3,2) \end{cases}$$



Notice: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$

$\epsilon_{ijk} = -\epsilon_{jik}$

Important Cross Product notation

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk}\, a_j\, b_k \qquad (3.1)$$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\,\hat{i} + (a_3 b_1 - a_1 b_3)\,\hat{j} + (a_1 b_2 - a_2 b_1)\,\hat{k}$$

$\underline{i=1}$

$$\epsilon_{1jk}\, a_j\, b_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk}\, a_j b_k = a_2 b_3 - a_3 b_2$$

only nonzero values of ϵ_{1jk}

$\epsilon_{123} = 1$, $\epsilon_{132} = -1$

Relationship between ϵ_{ijk} and δ_{ij}

Prop.– $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Proof.– The above equation produces 81 different equations!

Only few of them are nontrivial. Let's analyze $\underline{\epsilon_{ijk}\epsilon_{klm}}$.

If $i=1$

If $j=2$
then $k=3$

If $l=1$
then $m=2$ and $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{123}\epsilon_{312} = 1$

If $l=2$
then $m=1$ and $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{123}\epsilon_{321} = -1$.

If $j=3$
then $k=2$

If $l=1$
then $m=3$ and $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{132}\epsilon_{213} = (-1)(-1) = 1$

If $l=3$
then $m=1$ and $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{132}\epsilon_{231} = (-1)(1) = -1$

Similar for $i=2,3$.

On the other hand, if $\underline{\underline{i=1, \text{ and } j=2}}$

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \delta_{1l}\delta_{2m} - \delta_{1m}\delta_{2l} =$$

$$= \begin{cases} 1, & \text{if } l=1,\ m=2 \\ -1, & \text{if } m=1,\ l=2 \end{cases}$$

If $\underline{\underline{i=1,\ j=3}}$

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \delta_{1l}\delta_{3m} - \delta_{1m}\delta_{3l} =$$

$$= \begin{cases} 1, & l=1,\ m=3 \\ -1, & l=3,\ m=1 \end{cases}$$

Similar for $i = 2, 3$.

Therefore,

$$\boxed{\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.} \quad (5.1)$$

Exercise:-

$$\left(\vec{a}\times(\vec{b}\times\vec{c})\right)_i \overset{\text{See (3.1)}}{=} \epsilon_{ijk}\, a_j\, (b\times c)_k =$$

$$= \epsilon_{ijk}\, a_j\, \epsilon_{klm}\, b_l c_m =$$

$$= \epsilon_{ijk}\,\epsilon_{klm}\, a_j b_l c_m \overset{(5.1)}{=} \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) a_j b_l c_m$$

$$= a_j b_i c_j - a_j b_j c_i =$$

$$= (\vec{a}\cdot\vec{c})\, b_i - (\vec{a}\cdot\vec{b})\, c_i$$

$$\therefore \quad \boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}.}$$

Gradient, Divergence and Curl in Suffix notation and Combination of them.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

$$(\nabla f)_i = \frac{\partial f}{\partial x_i}$$

$$\nabla \cdot \vec{u} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

is given by $\dfrac{\partial u_i}{\partial x_i}$ in suffix notation.

$$\nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\hat{\imath} + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\hat{\jmath}$$
$$+ \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\hat{k}$$

then $$\left(\nabla \times \vec{u}\right)_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = \epsilon_{1jk}\frac{\partial u_k}{\partial x_j}$$

and in general
$$(\nabla \times \vec{u})_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

Applications: Show that

a) $\nabla \times (\nabla f) = 0$, b)

Proof:-

a) $$\left[\nabla \times (\nabla f)\right]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\nabla f)_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right)$$

if $i=1$
$$\epsilon_{1jk}\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \epsilon_{123}\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_3}\right) + \epsilon_{132}\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial x_2}\right)$$
$$= \frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2} = 0.$$

Similar for $i = 2, 3$. ✓

b) (Hwk 1 problem ①ⓐ) Using suffix notation.

$$\left[\nabla\times(\nabla\times\vec{A})\right]_i = \epsilon_{ijk}\frac{\partial(\nabla\times A)_k}{\partial x_j} = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial A_m}{\partial x_l}\right)$$

$$= \epsilon_{ijk}\epsilon_{klm}\frac{\partial^2 A_m}{\partial x_j\cdot\partial x_l} = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial^2 A_m}{\partial x_j\cdot\partial x_l}$$

$$= \frac{\partial^2 A_j}{\partial x_j\,\partial x_i} - \frac{\partial^2 A_i}{\partial x_j\cdot\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial A_j}{\partial x_j}\right) - (\nabla^2 A)_i$$

$$= \frac{\partial}{\partial x_i}(\nabla\cdot\vec{A}) - (\nabla^2 A)_i = \left[\nabla(\nabla\cdot\vec{A})\right]_i - (\nabla^2 A)_i$$

$$\therefore \boxed{\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}.}$$