Appendix

Mathematical Relationships

A.1 VECTOR IDENTITIES

A is a vector defined as

$$\mathbf{A} = \mathbf{e}_1 A_1 + \mathbf{e}_2 A_2 + \mathbf{e}_3 A_3$$

where $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors in the coordinate directions and A_1, A_2, and A_3 are the components of the vector. A scalar is denoted as Φ. ∇ is the operator "del".

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \qquad \textbf{A.1}$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad \textbf{A.2}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \qquad \textbf{A.3}$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \qquad \textbf{A.4}$$

$$(\mathbf{A}+\mathbf{B})\times\mathbf{C} = (\mathbf{A}\times\mathbf{C}) + (\mathbf{B}\times\mathbf{C}) \qquad \textbf{A.5}$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C}) \qquad \textbf{A.6}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad \textbf{A.7}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \qquad \textbf{A.8}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \qquad \textbf{A.9}$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{B}[\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})] - \mathbf{A}[\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})] \\ &= \mathbf{C}[\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})] - \mathbf{D}[\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})]\end{aligned} \qquad \textbf{A.10}$$

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi \qquad \textbf{A.11}$$

$$\nabla^2\mathbf{A} = (\nabla\cdot\nabla)\mathbf{A} \qquad \textbf{A.12}$$

$$\nabla\cdot\nabla\times\mathbf{A} = 0 \qquad \textbf{A.13}$$

$$\nabla \times \nabla\Phi = 0 \qquad \text{A.14}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \qquad \text{A.15}$$

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = \nabla\left(\frac{|\mathbf{A}|^2}{2}\right) - \mathbf{A} \times (\nabla \times \mathbf{A}) \qquad \text{A.16}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) \qquad \text{A.17a}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \qquad \text{A.17b}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \qquad \text{A.18}$$

A.2 VECTOR INTEGRALS

Listed below are the more common vector integral theorems.

(a) Stokes' Theorem.

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \qquad \text{A.19}$$

where the curve C defines the surface S. Fig. A-1 below illustrates the surface. The sense of the line integral is counterclockwise with respect to the vector $d\mathbf{S}$.

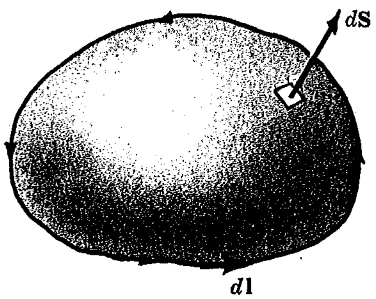

Fig. A-1

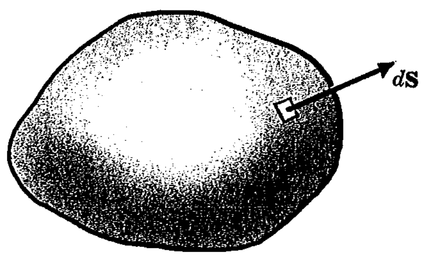

Fig. A-2

(b) Gauss' Theorem or Divergence Theorem.

$$\int_V \nabla \cdot \mathbf{A}\, dV = \int_S \mathbf{A} \cdot d\mathbf{S} \qquad \text{A.20}$$

where the surface S defines the volume V. Fig. A-2 above shows this surface and volume.

(c) Green's Theorems.

If a vector $\mathbf{A}$ is derived from a potential Φ as

$$\mathbf{A} = \nabla\Phi$$

then the line integral of the vector $\mathbf{A}$ can be expressed as

$$\int_A^B \mathbf{A} \cdot d\mathbf{l} = \int_A^B d\Phi \qquad \text{A.21}$$