1.1 Leibniz's Rule

$F(x,y)$ integrable of y for each value of x

$\frac{\partial F}{\partial x}$ exists and is continuous in $R : [a(x), b(x)] \times [a(x), b(x)]$
$[x_0, x_1] \times [x_0, x_1]$

$a(x)$, $b(x)$ are diff. on $[x_0, x_1]$

$$\Rightarrow \quad \frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} F_x(x,y)\,dy + F(x,b(x))\,b'(x) - F(x,a(x))\,a'(x).$$

Define $G(x,y)$ as

$$\boxed{F(x,y) = \frac{\partial G}{\partial y}(x,y)}$$

$$\Rightarrow \int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} \frac{\partial G}{\partial y}(x,y)\,dy \overset{F.T.C.}{=} G(x,b(x)) - G(x,a(x))$$

$$\therefore \frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \frac{d}{dx}\int_{a(x)}^{b(x)} \frac{\partial G}{\partial y}(x,y)\,dy = \frac{\partial G}{\partial x}(x,b(x)) + \frac{\partial G}{\partial y}(\;)\,b'(x) - \frac{\partial G}{\partial x}(x,a(x)) - \frac{\partial G}{\partial y}(\;)\,a'(x)$$

$$= \frac{\partial G}{\partial x}(x,b(x)) - \frac{\partial G}{\partial x}(x,a(x)) + \frac{\partial G}{\partial y}(\;)\,b'(x) - \frac{\partial G}{\partial y}(\;)\,a'(x)$$

$$= \underbrace{\int_{a(x)}^{b(x)} \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right)(x,y)\,dy}_{= \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right)(x,y)\,dy \;=\; \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}(F)(x,y)\,dy} + F(x,b(x))\,b'(x) - F(x,a(x))\,a'(x).$$

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1.2

2 #8

$$u(x) = x + \mu \int_0^x (x-y)\, u(y)\, dy$$

Neumann Series:

$u_0(x) \equiv 0$

$$u_1(x) = x + \mu \int_0^x (x-y)\, u_0(y)\, dy = x + \mu \int_0^x (x-y)\, 0\, dy = x.$$

$$u_2(x) = x + \mu \int_0^x (x-y)\, u_1(y)\, dy = x + \mu \int_0^x (x-y)\, y\, dy = x + \mu \int_0^x xy - y^2\, dy$$

$$= x + \mu \left[\frac{xy^2}{2} - \frac{y^3}{3} \right]\Bigg|_0^x = x + \mu \left[\frac{x^3}{2} - \frac{x^3}{3} \right] = x + \mu \frac{x^3}{6}$$

$$u_3(x) = x + \mu \int_0^x (x-y)\, u_2(y)\, dy =$$

$$= x + \mu \int_0^x (x-y)\left(y + \mu \frac{y^3}{6}\right) dy =$$

$$= x + \mu \int_0^x \left[xy + \mu \frac{xy^3}{6} - y^2 - \mu \frac{y^4}{6} \right] dy = x + \mu \left[\frac{xy^2}{2} + \mu \frac{xy^4}{24} - \frac{y^3}{3} - \mu \frac{y^5}{30} \right]_0^x =$$

$$= x + \mu \left[\frac{x^3}{2} + \mu \frac{x^5}{24} - \frac{x^3}{3} - \mu \frac{x^5}{30} \right]$$

$$= x + \mu \left[\frac{x^3}{6} + \mu \frac{x^5}{120} \right]$$

⑤ 15

#11

$$\int_0^x y\,u(y)\,dy - \lambda u(x) = f(x), \qquad 0 \le x \le 1.$$

$f'(x)$ exists., $\boxed{\lambda \neq 0}$ $\qquad \boxed{u(0) = \frac{f(0)}{\lambda}}$

$$\frac{d}{dx}\left[\quad = \quad\right] \Rightarrow$$

$$x\,u(x)\,x'^{\,1} - 0 - \lambda u'(x) = f'(x).$$

$$\Rightarrow \quad u'(x) - \frac{x}{\lambda}u(x) = -\frac{1}{\lambda}f'(x)$$

$$\text{I.F.} = e^{-\int \frac{x}{\lambda}dx} = e^{-\frac{x^2}{2\lambda}}$$

$$\frac{d}{dx}\left[u\,e^{-\frac{x^2}{2\lambda}}\right] = -\frac{1}{\lambda}e^{-\frac{x^2}{2\lambda}}f'(x).$$

$$\int_0^x \quad u(x)\,e^{-\frac{x^2}{2\lambda}} - u(0)\,e^0 = -\frac{1}{\lambda}\int_0^x e^{-\frac{\bar{x}^2}{2\lambda}}f'(\bar{x})\,d\bar{x}$$

Now, $$\int_0^x e^{-\frac{\bar{x}^2}{2\lambda}}f'(\bar{x})\,d\bar{x} = e^{-\frac{x^2}{2\lambda}}f(x) - f(0) - \int_0^x \frac{-2\bar{x}}{2\lambda}e^{-\frac{\bar{x}^2}{2\lambda}}f(\bar{x})\,d\bar{x}$$

$$\frac{f(0)}{\lambda} = u(0)$$

$$\boxed{u(x) = u(0)\,e^{\frac{x^2}{2\lambda}} + \frac{e^{\frac{x^2}{2\lambda}}}{\lambda^2}\int_0^x \bar{x}\,e^{-\frac{\bar{x}^2}{2\lambda}}f(\bar{x})\,d\bar{x}}$$

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(10)

1.10 Investigate Solvability.

#17 $$u(x) = \sin x + 3\int_0^{\pi}(x+y)\,u(y)\,dy$$

answer: $$\int_0^{\pi}(x+y)u(y)dy - \frac{1}{3}u(x) = -\frac{1}{3}\sin x.$$

$$\boxed{Ku - \frac{1}{3}u = -\frac{1}{3}\sin x} \qquad (1)$$

Kernel: $k(x,y) = x+y$ is separable $\boxed{\begin{array}{ll}\alpha_1(x) = x, & \beta_1(y) = 1\\ \alpha_2(x) = 1, & \beta_2(y) = y\end{array}}$

$$= \sum_{j=1}^{2}\alpha_j(x)\beta_j(y)$$

Strategy: Reduce (1) to a linear system.

$$\boxed{\left(A - \frac{1}{3}I\right)\vec{C} = \vec{F}} \qquad (2)$$

where $A = \left((\beta_i, \alpha_j)\right)_{2\times 2}$ $\quad \vec{F} = \begin{pmatrix}(-\frac{1}{3}\sin x, 1)\\ (-\frac{1}{3}\sin x, x)\end{pmatrix}$

and $\vec{C} = \begin{pmatrix}(u(x), 1)\\ (u(x), x)\end{pmatrix}$

and prove $\lambda = -\frac{1}{3}$ is not an eigenvalue of A.

Then thm. 1.2 in page 249 Logan's book guarantee that there is a unique solution for (1) given by

$$u(x) = +3\left(+\frac{1}{3}\sin x + \sum_{j=1}^{2}\alpha_j(x)c_j\right) \qquad (3)$$

where $\vec{C} = \binom{c_1}{c_2}$ is the unique solution of (2).

Matrix A:

$$(\beta_1, \alpha_1) = \int_0^\pi x\,dx = \frac{\pi^2}{2}, \qquad (\beta_2, \alpha_1) = \int_0^\pi x^2 dx = \frac{\pi^3}{3}$$

$$(\beta_1, \alpha_2) = \int_0^\pi dx = \pi, \qquad (\beta_2, \alpha_2) = \int_0^\pi x\,dx = \frac{\pi^2}{2}$$

Then,

$$A = \begin{pmatrix} \pi^2/2 & \pi \\ \pi^3/3 & \pi^2/2 \end{pmatrix}$$

Charact. polyn.:

$$|A - \lambda I| = \left(\frac{\pi^2}{2} - \lambda\right)^2 - \frac{\pi^4}{3} = 0$$

or $\lambda^2 - \pi^2\lambda + \frac{\pi^4}{4} - \frac{\pi^4}{3} = 0 \Leftrightarrow \lambda^2 - \pi^2\lambda - \frac{\pi^4}{12} = 0$

$$\Rightarrow \quad \lambda = \frac{\pi^2 \pm \sqrt{\pi^4 + \frac{\pi^4}{3}}}{2} = \frac{\pi^2}{2} \pm \frac{\sqrt{\frac{4\pi^4}{3}}}{2} = \frac{\pi^2}{2} \pm \frac{\pi^2}{\sqrt{3}}.$$

or $\lambda = \frac{\pi^2}{2} \pm \frac{\sqrt{3}}{3}\pi^2$ $\to$ $\lambda_1 = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right)\pi^2$, $\lambda_2 = \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right)\pi^2$

Thus, $\lambda = -1/3$ is not an eigenvalue.

and (1) has a unique solution.

$$\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad F_1 = \int_0^\pi -\frac{1}{3}\sin x\,dx = +\frac{1}{3}\cos x\Big|_0^\pi = -\frac{2}{3}.$$

$$F_2 = \int_0^\pi \left(-\frac{1}{3}\sin x\right) x\,dx = -\frac{1}{3}\left(\sin x - x\cos x\right)\Big|_0^\pi = -\frac{\pi}{3}$$

Therefore, associated linear system:

$$\begin{pmatrix} \pi^2/2 - 1/3 & \pi \\ \pi^3/3 & \pi^2/2 - 1/3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -\pi/3 \end{pmatrix}$$

Once $\vec{c}$ is obtained $u(x)$ is obtained from (3).

1.7 Does the operator

#14

$$Ku(x) = \int_0^{\pi} \sin x \sin 2y \, u(y) dy$$

have any eigenvalues?

Answ: Kernel $k(x,y) = \sin x \sin 2y$

is separable with $\alpha_1(x) = \sin x$, $\beta_1(y) = \sin 2y$.

Therefore, $\underline{\underline{\lambda = 0}}$ is always an eigenvalue. We can find any other eigenvalue by finding the eigenvalues of the associated linear system.

see class notes.

Go to next page

$$Ku(x) = \int_0^{\pi} \sin x \sin 2y \, u(y) dy = \sin x \int_0^{\pi} \sin 2y \, u(y) \, dy$$

$$= \sin x \, C$$

where $C \equiv (\sin 2x, u(x))$.

$\underline{\underline{EVP}}$, $u \neq 0$ s.t. $Ku = \lambda u \iff \sin x \, C = \lambda u(x)$

multiplying by $\sin 2x$ $\int_a^b dx$

$$(\sin 2x, \sin x) \, C = \lambda (u, \sin 2x) = \lambda C$$

Associated linear syst. $AC - \lambda C = 0 \Rightarrow (A - \lambda) C = 0$, $C \neq 0$

$\Downarrow$

$\lambda = A = 0 \Rightarrow \underline{\underline{\lambda = 0}}$

$$A \equiv \int_0^{\pi} \sin 2x \sin x \, dx = 0$$

only eigenvalue!

$\Rightarrow C$ is arbitrary, in particular $\boxed{C=1}$ is an eigenvector

Back to our F.I Equ. EVP. with $\lambda = 0$

$$Ku = \lambda \overset{=0}{u} \iff (\sin 2y, u(y)) \sin x = \overset{=0}{\lambda} u(x) = O(x)$$

$$\Rightarrow (\sin 2y, u(y)) \overset{\text{needs to be}}{=} 0$$

It means any function $u(y) \perp \sin 2y$ will be an eigenfunction for $\lambda = 0$ $\{\sin my\}_{m=1}^{\infty}$, $m \neq 2$. Therefore, there are infinitely many lin. independent eigenfns. associated to $\lambda = 0$.

$$Ku = 0 \iff$$

$$Ku_{(x)} = \int_0^{\pi} \sin x \sin 2y\, u(y)\, dy = 0$$

$$\Rightarrow \sin x \int_0^{\pi} \sin 2y\, u(y)\, dy = 0$$

It's enough that $u(y) \perp \sin(2y)$