Integral Equations.

Unknown function is under integral sign.

Examples:

Fredholm equation:

$$\int_a^b K(x,y)\,u(y)\,dy + \alpha(x)\,u(x) = f(x), \qquad a \le x \le b. \quad (1)$$

Volterra equation:

$$\int_a^{x} k(x,y)\,u(y)\,dy + \alpha(x)\,u(x) = f(x), \quad a \le x \le b \quad (2)$$

$u(x,y)$: unknown function.

$K(x,y)$: Kernel

If $f(x) \equiv 0$ equations are called homogeneous otherwise nonhomogeneous.

If $\alpha(x) \equiv 0$ equations are said to be of the first kind, otherwise they are of 2^{rd} kind.

Integral Operator:

$$K[u](x) = \int_a^b K(x,y)\, u(y)\, dy.$$

Then (1) can be written as

$$K[u] + \alpha u = f$$

Eigenvalues: Any μ such that there exists $u(x) \not\equiv 0$

and $\quad K[u] = \mu u$

Spectrum = { set of eigenvalues }.

Inner Product and Norm: $\quad u, v \in C[a,b]$ (or more general $u, v \in L_2[a,b]$)

Inner Product: $\quad (u, v) = \int_a^b u(x)\, \overline{v(x)}\, dx.$

Norm: $\quad \|u\| = \int_a^b |u(x)|^2 dx$

Example of a Volterra Equation:

Inventory control Problem:

a: Amount of goods initially purchased ($t=0$).

$u(t)$: Rate at which goods are purchased.

$K(t)$: Percentage of goods that remains unsold at time t, after purchasing the goods.

It's a fraction. For example 30% → 0.03.

Condition: Want stocks to remain constant.

MATH Model: Question: ~~At what rate the goods need to be purchased~~ for the stock to remain constant?

$$a K(t) + \int_0^t K(t-\tau)\, u(\tau)\, d\tau = a.$$

Why? Consider the interval $[\tau, \tau+\Delta\tau]$

a) Amount purchased in time interval $\Delta\tau$:

$$\underbrace{u(\bar{t})}_{\text{Average in that time interval.}} \Delta\tau, \qquad \bar{t} \in [\tau, \tau+\Delta\tau].$$

b) Portion of (a) remaining unsold at time t:

$$K(t-\tau)\, u(\bar{t})\, \Delta\tau,$$

If $\Delta\tau \to 0$ $\quad \Delta\tau \to d\tau$

$$K(t-\tau)\,u(\tau)\,\Delta\tau \to K(t-\tau)\,u(\tau)\,d\tau.$$

we get the portion remaining unsold at the time instant $t=\tau$. Therefore, adding $\int_0^t d\tau$ we will get

the total portion remaining unsold at time t.

for the process

$$\int_0^t K(t-\tau)\,u(\tau)\,d\tau$$

This is not complete until we add the portion of the initial amount "a" that remains unsold at t.

$\underline{a\,K(t)}$

i.e.,

<u>Total Portion Remaining Unsold:</u>

status remains constant

$$a\,K(t) + \int_0^t K(t-\tau)\,u(\tau)\,d\tau = a.$$

Laplace Transform applied to a special Volterra I. Equ.

Consider

$$u(x) = f(x) + \int_0^x K(x-y)\, u(y)\, dy. \qquad (5.1)$$

Integral equation of convolution type.

$$\int_0^x K(x-y)\, u(y)\, dy = K * u.$$

We know

Laplace transf.

$$\mathcal{L}\{K * u\} = \mathcal{L}\{K\}\, \mathcal{L}\{u\}$$

Taking L.T. of (5.1).

$$\mathcal{L}\{u\} = \mathcal{L}\{f\} + \mathcal{L}\{K\}\mathcal{L}\{u\}$$

$$\Rightarrow \qquad \mathcal{L}\{u\} = \frac{\mathcal{L}\{f\}}{1 - \mathcal{L}\{K\}}$$

In particular, if we want to solve

$$u(x) = x - \int_0^x \overset{K(x-y)}{\overset{\shortparallel}{(x-y)}}\, u(y)\, dy$$

$\Rightarrow \quad K(x) = x.$

Applying L.T.

$$\mathcal{L}\{u\} = \mathcal{L}\{x\} - \mathcal{L}\{K\}\,\mathcal{L}\{u\}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}\,\mathcal{L}\{u\}$$

$$\Rightarrow \quad \mathcal{L}\{u\} = \frac{1/s^2}{1 + 1/s^2} = \frac{1}{s^2+1}$$

$\downarrow \mathcal{L}^{-1} \qquad\qquad \downarrow \mathcal{L}^{-1}$

$$\Rightarrow \quad \boxed{u(x) = \sin(x)}$$

<u>Reformulation of IVP as a Volterra Int. Equs.</u>

$$\boxed{u'(x) = f(x,u), \quad u(x_0) = u_0}$$

$$\int_{x_0}^{x} u'(y)\, dy = \int_{x_0}^{x} f(y, u(y))\, dy$$

$\downarrow$ FTC.

$$\boxed{u(x) = u(x_0) + \int_{x_0}^{x} f(y, u(y))\, dy}$$

Second order IVP (ODE) $\longrightarrow$ Volterra Integ. Equ.

Lemma.– $f(x)$ continuous for $x \geq a$. Then

$$\int_a^x \left(\int_a^s f(y)\, dy \right) ds = \int_a^x f(y)\,(x-y)\, dy$$

Proof.–

$$\int_a^x \Big(\overbrace{\int_a^s f(y)\, dy}^{=F(s)} \Big) ds = \int_a^x \underset{f}{\underset{\downarrow}{F(s)}}\, \underset{g'}{\underset{\downarrow}{ds}} \overset{\text{I.P.P.}}{=}$$

$$s\,F(s)\Big|_a^x - \int_a^x s\,F'(s)\, ds = x\,F(x) - a\,\cancelto{0}{F(a)} - \int_a^x s\,F'(s)\, ds$$

$$= x \int_a^x f(y)\, dy - \int_a^x s\, f(s)\, ds = x \int_a^x f(y)\, dy - \int_a^x y\, f(y)\, dy$$

or

$$\int_a^x \int_a^s f(y)\, dy\, ds = \int_a^x f(y)\,(x-y)\, dy$$

<u>Example:</u> Consider

$$\begin{cases} u'' + p(x)u' + q(x)u = f(x), & x > a \\ u(a) = u_0, & u'(a) = u_1 \end{cases}$$

$$u'' = -p(x)u' - q(x)u + f(x).$$

$\int_a^x dy$

$$u'(x) - u'(a) = -\int_a^x p(y)u'(y)\,dy - \int_a^x \left(q(y)u(y) - f(y)\right)dy$$

$$\Rightarrow u'(x) - u'(a) \overset{\text{I.P.P.}}{=} -p(x)u(x) + p(a)u(a) - $$

$$+ \int_a^x p'(y)u(y)\,dy - \int_a^x \left[q(y)u(y) - f(y)\right]dy$$

$\therefore$

$$u'(x) - u_1 = -p(x)u(x) + p(a)u_0 - \int_a^x \left\{\left[q(y) - p'(y)\right]u(y) - f(y)\right\}dy$$

$\int_a^x dy$

$$u(x) - \underset{}{\overset{=u_0}{u(a)}} - u_1(x-a) = -\int_a^x p(x)u(x)\,dx + p(a)u_0(x-a) -$$

$$- \int_a^x \left(\int_a^s \left\{\left[q(y) - p'(y)\right]u(y) - f(y)\right\}dy\right)ds$$

$\therefore$ Using lemma.

$$u(x) = u_0 + (p(a) + u_1)(x-a) - \int_a^x p(x)u(x)\,dx$$

$$- \int_a^x [q(y) - p'(y)]\, u(y)\,(x-y)\,dy$$

$$+ \int_a^x f(y)\,(x-y)\,dy$$

or

$$\boxed{u(x) = u_0 + [p(a) + u_1](x-a) - \int_a^x \Big[p(y) + [q(y) - p'(y)](x-y)\Big] u(y)\,dy + \int_a^x f(y)(x-y)\,dy.}$$

This equation is of the form of a Volterra equ. of the second kind.

$$u(x) = \int_a^x K(x,y)\, u(y)\,dy + F(x).$$