Initial Value problem for first order ODE and their representation as a nonlinear Volterra equation.

Initial Value problem governed by a nonlinear first order ODE:

$$\begin{cases} u'(x) = f(x, u(x)) & (1) \\ u(x_0) = u_0 & (2) \end{cases}$$

Integrating (1) along the interval $[x_0, x]$

$$\int_{x_0}^{x} u'(y)\,dy = \int_{x_0}^{x} f(y, u(y))\,dy$$

Applying Fund. thm. of Calculus:

$$u(x) - u(x_0) = \int_{x_0}^{x} f(y, u(y))\,dy$$

or $$\boxed{u(x) = \int_{x_0}^{x} f(y, u(y))\,dy + u_0}$$

Nonhomogeneous Volterra equation of the second kind.

If f and $\frac{\partial f}{\partial u}$ are continuous on certain rectangle $R: \{(x,u): |x| \leq a, |u| \leq b\}$.

the integral equation:

$$u(x) = u_0 + \int_{x_0}^{x} f(y, u(y))\, dy \qquad (11.1)$$

can be solved by mean of Picard iteration (fixed point iteration).

Procedure consists of starting with an initial approximation $\phi_0(x)$ for the solution $\phi(x)$ of (11.1). and from this initial approximation generate a sequence of successive approximations: $\phi_1(x), \phi_2(x), \ldots$ as follows:

$$\phi_{n+1}(x) = u_0 + \int_a^x f(y, \phi_n(y))\, dy \qquad n = 0, 1, 2 \ldots$$

It can be proved that (11.1) has a unique solution $\phi(x)$ for x in certain interval and that the sequence

$$\phi_n(x) \xrightarrow[n \to \infty]{} \phi(x)$$

In the following pages from Boyce - D'prima's book,

the IVP

$$\begin{cases} y'(t) = 2t\,(1+y(t)) \\ y(0) = 0 \end{cases}$$

is considered.

It's Solution can be easily obtained by Separation of variables

$$\boxed{\phi(t) = e^{t^2} - 1}$$

By using Picard iteration or "Succesive approximations"

the following Sequence is generated

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!}$$

which clearly Converges to

$$\phi(t) = e^{t^2} - 1 ..$$

Similarly, ϕ_2 is obtained from ϕ_1:

$$\phi_2(t) = \int_0^t f[s, \phi_1(s)]\, ds, \tag{6}$$

and, in general,

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds. \tag{7}$$

In this manner we generate the sequence of functions $\{\phi_n\} = \phi_0, \phi_1, \ldots, \phi_n, \ldots$. Each member of the sequence satisfies the initial condition, but in general none satisfies the differential equation. However, if at some stage, say for $n = k$, we find that $\phi_{k+1}(t) = \phi_k(t)$, then it follows that ϕ_k is a solution of the integral equation (3). Hence ϕ_k is also a solution of the initial value problem (2), and the sequence is terminated at this point. In general, this does not occur, and it is necessary to consider the entire infinite sequence.

To establish Theorem 2.8.1 four principal questions must be answered:

1. Do all members of the sequence $\{\phi_n\}$ exist, or may the process break down at some stage?
2. Does the sequence converge?
3. What are the properties of the limit function? In particular, does it satisfy the integral equation (3), and hence the initial value problem (2)?
4. Is this the only solution, or may there be others?

We first show how these questions can be answered in a specific and relatively simple example, and then comment on some of the difficulties that may be encountered in the general case.

EXAMPLE 1

Solve the initial value problem

$$y' = 2t(1 + y), \qquad y(0) = 0. \tag{8}$$

by the method of successive approximations.

Note first that if $y = \phi(t)$, then the corresponding integral equation is

$$\phi(t) = \int_0^t 2s[1 + \phi(s)]\, ds. \tag{9}$$

If the initial approximation is $\phi_0(t) = 0$, it follows that

$$\phi_1(t) = \int_0^t 2s[1 + \phi_0(s)]\, ds = \int_0^t 2s\, ds = t^2. \tag{10}$$

Similarly,

$$\phi_2(t) = \int_0^t 2s[1 + \phi_1(s)]\, ds = \int_0^t 2s[1 + s^2]\, ds = t^2 + \frac{t^4}{2} \tag{11}$$

and

$$\phi_3(t) = \int_0^t 2s[1 + \phi_2(s)]\, ds = \int_0^t 2s\left[1 + s^2 + \frac{s^4}{2}\right] ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2 \cdot 3}. \tag{12}$$

Equations (10), (11), and (12) suggest that

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} \tag{13}$$

for each $n \geq 1$, and this result can be established by mathematical induction. Equation (13) is certainly true for $n = 1$; see Eq. (10). We must show that if it is true for $n = k$, then it also holds for $n = k + 1$. We have

$$\begin{aligned} \phi_{k+1}(t) &= \int_0^t 2s[1 + \phi_k(s)]\, ds \\ &= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2!} + \cdots + \frac{s^{2k}}{k!}\right) ds \\ &= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2k+2}}{(k+1)!}, \end{aligned} \tag{14}$$

and the inductive proof is complete.

A plot of the first four iterates, $\phi_1(t), \ldots, \phi_4(t)$ is shown in Figure 2.8.1. As k increases, the iterates seem to remain close over a gradually increasing interval, suggesting eventual convergence to a limit function.

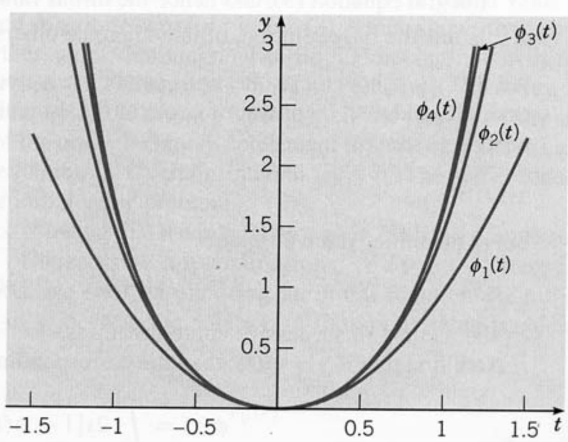

FIGURE 2.8.1 Plots of $\phi_1(t), \ldots, \phi_4(t)$ for Example 1.

It follows from Eq. (13) that $\phi_n(t)$ is the nth partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}; \tag{15}$$

hence $\lim_{n\to\infty} \phi_n(t)$ exists if and only if the series (15) converges. Applying the ratio test, we see that, for each t,

$$\left|\frac{t^{2k+2}}{(k+1)!}\,\frac{k!}{t^{2k}}\right| = \frac{t^2}{k+1} \to 0 \quad \text{as} \quad k \to \infty; \tag{16}$$

thus the series (15) converges for all t, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$. Further, since the series (15) is a Taylor series, it can be differentiated or integrated term by term as long as t remains within the interval of convergence, which in this case is the entire t-axis. Therefore, we can verify by direct computation that $\phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$ is a solution of the integral equation (9). Alternatively, by substituting $\phi(t)$ for y in Eqs. (8), we can verify that this function satisfies the initial value problem. In this example it is also possible, from the series (15), to identify ϕ in terms of elementary functions, namely, $\phi(t) = e^{t^2} - 1$. However, this is not necessary for the discussion of existence and uniqueness.

Explicit knowledge of $\phi(t)$ does make it possible to visualize the convergence of the sequence of iterates more clearly by plotting $\phi(t) - \phi_k(t)$ for various values of k. Figure 2.8.2 shows this difference for $k = 1, \ldots, 4$. This figure clearly shows the gradually increasing interval over which successive iterates provide a good approximation to the solution of the initial value problem.

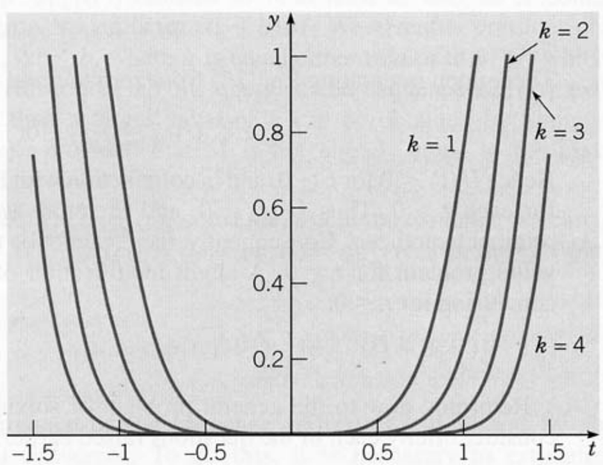

FIGURE 2.8.2 Plots of $\phi(t) - \phi_k(t)$ for Example 1 for $k = 1, \ldots, 4$.

Finally, to deal with the question of uniqueness, let us suppose that the initial value problem has two solutions ϕ and ψ. Since ϕ and ψ both satisfy the integral equation (9), we have by subtraction that

$$\phi(t) - \psi(t) = \int_0^t 2s[\phi(s) - \psi(s)]\, ds.$$

Taking absolute values of both sides, we have, if $t > 0$,

$$|\phi(t) - \psi(t)| = \left| \int_0^t 2s[\phi(s) - \psi(s)]\, ds \right| \le \int_0^t 2s|\phi(s) - \psi(s)|\, ds.$$