Consider the nonhomogeneous Volterra equation

$$\boxed{u(x) = \int_a^x K(x,y)\, u(y)\, dy + f(x)} \qquad (1)$$

by defining

$$Ku(x) \equiv \int_a^x K(x,y)\, u(y)\, dy$$

Equ. (1) can be rewritten as

$$\boxed{u = Ku + f} \qquad (2)$$

Picard iteration or Successive approximations for equ. (2) can be defined as

$$u_{n+1} = Ku_n + f\,, \qquad n = 0, 1, 2, \ldots \qquad (3)$$

To start choose an initial approximation $u_0(x)$ following "certain criteria" (to be determined by theorem) and generate

$$u_1 = Ku_0 + f$$

$$\vdots$$

$$u_{n+1} = Ku_n + f.$$

Theorem.- For each forcing function $f \in C[a,b]$ the Volterra integral equation of the second kind

$$u(x) - \int_a^x K(x,y)\,u(y)\,dy = f(x) \qquad (1) \text{ or } (2)$$

with continuous kernel K on the rectangle $[a,b]\times[a,b]$ has a unique solution $u \in C[a,b]$.

Proof.- Strategy

1) First construct successive approximations $\{u_j\}_{j=0}^{n+1}$ and show that

$$u_n(x) \xrightarrow[n\to\infty]{} \hat{u}(x) = \sum_{j=0}^{\infty} \left(K^j f\right)(x) \qquad (4)$$

where $\displaystyle Kf \equiv \int_a^x K(x,y)\,f(y)\,dy$

For (4) to make sense, we need to prove first that

$$\left(\sum_{j=0}^{\infty} K^j f\right)(x) \text{ converges.}$$

2) Then, we will prove that $\hat{u}(x)$ obtained in (4) satisfies (1).

3) We will prove that the solution $u(x)$ of (1) is unique.

Therefore, $u(x) = \hat{u}(x)$.

<u>Proof.</u>- 1) Construction of Successive approximations

$$u_1 = Ku_0 + f$$

$$u_2 = Ku_1 + f = K[Ku_0 + f] + f = K^2 u_0 + Kf + f.$$

$$u_3 = Ku_2 + f = K[K^2 u_0 + Kf + f] + f = K^3 u_0 + K^2 f + Kf + f$$

$$= K^3 u_0 + \sum_{j=0}^{2} K^j f$$

$$\vdots$$

$$\boxed{u_{n+1} = \ldots = K^{n+1} u_0 + \sum_{j=0}^{n} K^j f} \qquad (5)$$

Clearly, (5) makes sense when $n \to \infty$ **if**

$S = \sum_{j=0}^{\infty} K^j f$ converges. and $K^{n+1} u_0$ also converges when $n \to \infty$

It's easy to prove that S is absolutely convergent.

In fact,

$$|Kf(x)| = \left|\int_a^x k(x,y)\, f(y)\, dy\right| \leq MC(x-a)$$

where $M \equiv \max_{a \leq x,y \leq b} |K(x,y)|$, $C \equiv \max_{a \leq x \leq b} |f(y)|$

$$\therefore \quad |K^2 f(x)| = \left| \int_a^x k(x,y)\,(Kf)(y)\,dy \right| \leq$$

$$\leq M\,(MC) \int_a^x (y-a)\,dy = M^2 C\,\frac{(x-a)^2}{2}$$

$$\vdots$$

$$|K^n f(x)| \leq M^n C\,\frac{(x-a)^n}{n!} \leq C\,\frac{M^n (b-a)^n}{n!}, \quad \text{for all } x \in [a,b]$$

$$\therefore \quad \sum_{j=0}^{\infty} |K^j f(x)| \leq C \sum_{j=0}^{\infty} \frac{M^n (x-a)^n}{n!} \leq C \sum_{j=0}^{\infty} \frac{M^n (b-a)^n}{n!},$$

for all $x \in [a,b]$

This last series is convergent. In fact, applying the radio test

$$\lim_{n\to\infty} \frac{\dfrac{M^{n+1}(b-a)^{n+1}}{(n+1)!}}{\dfrac{M^n (b-a)^n}{n!}} = \lim_{n\to\infty} \frac{M(b-a)}{n+1} = 0$$

Therefore, $\sum_{j=0}^{\infty} |K^j f(x)|$ Converges uniformly on $[a,b]$

Thus

$$U_n(x) = \sum_{j=0}^{n} K^j f(x) \longrightarrow \hat{u}(x), \quad \text{uniformly on } [a,b]$$

Also, $K^n u_0(x) \xrightarrow[n\to\infty]{} 0$ uniformly

we have already proved that

$$\left|K^n f(x)\right| \le \frac{C M^n (b-a)^n}{n!}, \quad \text{for all } x \in [a,b]$$

and for all $f \in C[a,b]$
Such that $C \equiv \max_{x\in[a,b]} |f(x)|$

Therefore, for $u_0 \in C[a,b]$

$$\left|K^n u_0(x)\right| \le \frac{d\, M^n (b-a)^n}{n!}, \quad \text{where } d \equiv \max_{x\in[a,b]} |u_0(x)|$$

This last term $\frac{d M^n (b-a)^n}{n!} \xrightarrow[n\to\infty]{} 0$

Since, we have already proved that

$$\sum_{n=0}^{\infty} \frac{M^n (b-a)^n}{n!}$$ converges, and as a consequence

$$\frac{M^n (b-a)^n}{n!} \xrightarrow[n\to\infty]{} 0 .$$

2) Prove that $\hat{u}(x)$ Satisfies (1), i.e.,

$$\boxed{\hat{u}(x) - \int_a^x K(x,y)\, \hat{u}(y)\, dy = f(x),} \quad \text{for } x \in [a,b]. \qquad (6)$$

From the definition of the Successive approximations

$$u_{n+1}(x) = K u_n + f \doteq \int_a^x K(x,y)\, u_n(y)\, dy + f(x)$$

Since $u_n(x) \underset{n\to\infty}{\longrightarrow} \hat{u}(x)$, uniformly on $[a,b]$

then $\displaystyle\int_a^x K(x,y)\, u_n(y)\, dy \longrightarrow \int_a^x K(x,y)\, \hat{u}(y)\, dy$, uniformly on $[a,b]$

In fact,

for any $\varepsilon > 0$ there exists N_ε s.t.

$$n \geq N_\varepsilon \Rightarrow |u_n(x) - \hat{u}(x)| < \varepsilon, \text{ for all } x \text{ on } [a,b]$$

Therefore,

if $n \geq N_\varepsilon \Rightarrow \displaystyle\left| \int_a^x K(x,y)\, u_n(y)\, dy - \int_a^x K(x,y)\, \hat{u}(y)\, dy \right| \leq$

$$\leq \int_a^x |K(x,y)|\, |u_n(y) - \hat{u}(y)|\, dy \leq M \varepsilon (x-a) \leq M(b-a)\varepsilon$$

$$\Rightarrow \int_a^x K(x,y)\, u_n(y)\, dy \underset{n\to\infty}{\longrightarrow} \int_a^x K(x,y)\, \hat{u}(y)\, dy.$$

Therefore, the left hand side of (6.2): $u_{n+1}(x) \xrightarrow[n\to\infty]{} \hat{u}(x)$

and the right hand side of 6.2): $\int_a^x k(x,y)u_n(y)dy + f \xrightarrow[n\to\infty]{} \int_a^x k(x,y)\hat{u}(y)\,dy + f$

As a result, equation (6) is satisfied by

$$\hat{u}(x) = \lim_{n\to\infty} u_n(x) = \sum_{j=0}^{\infty} K^j f$$

(3) Now, we will prove that $\hat{u}(x)$ is the only solution (uniqueness).

As usual, assume two solutions $u(x)$ and $v(x)$ then $w(x) \equiv u(x) - v(x)$ Satisfies the homogeneous Volterra integral equation

$$\boxed{w(x) - \int_a^x k(x,y)\,w(y)dy = 0,} \quad \text{for all } x \in [a,b]. \qquad (7)$$

$$|w(x)| = \left| \int_a^x k(x,y)\,w(y)\,dy \right| \le \int_a^x |k(x,y)|\,|w(y)|\,dy$$

$$\le MC\,(x-a)$$

Using this bound in (7)

$$|w(x)| \le \int_a^x |k(x,y)|\; MC\,(y-a)\,dy = C M^2 \frac{(x-a)^2}{2}$$

Continuing this process, we obtain

$$|w(x)| \le C\, M^n \frac{(x-a)^n}{n!} \le C\, \frac{M^n (b-a)^n}{n!} \xrightarrow[n\to\infty]{} 0$$

for all $x \in [a,b]$

$\Rightarrow$ $\boxed{w(x) \equiv 0}$ is the only soln. of $Ku = u$

$\Rightarrow$ $u(x) = v(x)$. or the solution for the nonhomogeneous linear Volterra equation is unique.

Eigenvalues of the operator

$$Ku \equiv \int_a^x k(x,y)\, u(y)\, dy.$$

Given $\lambda \neq 0$, is there any $u(x) \not\equiv 0$ s.t.

$$Ku = \lambda u\,?$$

Solving for u, $\quad u = \frac{1}{\lambda} Ku$

or $\quad u(x) = \frac{1}{\lambda} \int_a^x k(x,y)\, u(y)\, dy = \int_a^x \bar{k}(x,y)\, u(y)\, dy$

but only solution is $u(x) \equiv 0$. Therefore λ is not an eigenvalue.

<u>Bonus Hwk Problem</u> (10 points) a) Why do the proof of convergence for the successive approximations used for Volterra equations doesn't guarantee the convergence of them for Fredholm equations?

b) Do some research to find conditions of convergence for Fredholm int. eqns.