MASTER'S EXAMINATION IN MATHEMATICS
4 December 1999

INSTRUCTIONS.

MS Candidates: Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

MA Candidates: Answer a total of eight questions, with no restriction on category.

1. I. Basic Analysis
2. II. Basic Algebra
3. III. Analysis
4. IV. Advanced Algebra
5. V. Applied Mathematics
6. VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Do one of the following:

   (a) If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, show $\lim_{n\to\infty} a_n b_n = AB$. State clearly any assumptions.

   (b) Let (a_n) be a sequence such that $a_1 \leq a_2 \leq a_3 \leq \cdots$. If L is the least upper bound of the set $\{a_1, a_2, a_3, \cdots\}$, show that $\lim_{n\to\infty} a_n = L$.

   (c) Define a Cauchy sequence and show that a sequence of real numbers converges if and only if it is a Cauchy sequence.

2. Do one of the following.

   (a) If f is a continuous function on $[a, b]$, show $\int_a^b f(x)dx = f(c)(b - a)$ for some c, $a \leq c \leq b$.

   (b) If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), show that there is a number $c, a < c < b$ so that $f'(c)(b-a) = f(b) - f(a)$.

   (c) Let f be a real-valued function defined on an open iterval (a, b). If f has positive first derivative, show that f is increasing.

3. Do one of the following.

   (a) If f is a continuous function on the closed interval $[a, b]$, show f is integrable on $[a, b]$.

   (b) Let f be a bounded function on $[a, b]$ show f is integrable on $[a, b]$ if and only if for each $\epsilon > 0$, there is a partition P of [a,b] so that $U_f(P) - L_f(P) < \epsilon$, where $U_f(P)$ and $L_f(P)$ denote the upper and lower sum of f with respect to the partition P.

   (c) Evaluate the following iterated integral. You may consider reversing the order of integration.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_y^{\sqrt{\frac{\pi}{2}}} y^2 \sin x^2 dx\, dy$$

4. Do one of the following.

(a) State and prove the Ratio Test.

(b) State and prove the Integral Test.

(c) If $\sum_{k=0}^{\infty} a_k c^k$ is convergent and $|x| < |c|$, show that $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent.

II. BASIC ALGEBRA

1. Let $R_n[x]$ denote polynomials in x with coefficients in R and degree at most n. Define $D : R_n[x] \to R_n[x]$ by $D(f) = f'$. Show that D is a linear map from the vector space $R_n[x]$ to itself. What is the rank of D?

2. Prove that any two bases of a finite dimensional vector space have the same order.

3. How many nonisomorphic Abelian groups of order 72 are there?

4. Show that a subgroup H of a group G is normal if and only if H is the kernel of a homomorphism with domain G.

III. ANALYSIS

1.

IV. ADVANCED ALGEBRA

1. For a positive integer n, let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ denote the collection of residue classes modulo n.

   (a) Describe *all* group isomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_{15}$ to $\mathbb{Z}_6 \times \mathbb{Z}_5$ when both are regarded as additive abelian groups.

   (b) Which of the group homomorphisms in part (a) are also homomorphisms when $\mathbb{Z}_2 \times \mathbb{Z}_{15}$ and $\mathbb{Z}_6 \times \mathbb{Z}_5$ are regarded as rings in the standard way?

2. Let R be a commutative ring with $1 \neq 0$. Use the ring axioms to prove that $(-1)(-1) = 1$.

3. Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomial x^4+x^2+1.

4. If A is an invertible $n \times n$ matrix, prove that there is a polynomial $q(t)$ of degree $n-1$ such that $A^{-1} = q(A)$.

V. APPLIED MATHEMATICS

1. Find a function $g(x,t)$ such that

$$u(x) = \int_0^1 g(x,t) f(t) dt$$

solves the boundary value problem

$$\begin{aligned} u''(x) && = u(x) = f(x), \\ u(0) && = 0 = u'(1). \end{aligned}$$

2. For continuous real-valued functions on $[-1,1]$, the bilinear form

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx \tag{1}$$

defines an inner product.

(a) The Legendre polynomials are a set of polynomials orthogonal with respect to the inner product (1), and are obtained by performing the Gram-Schmidt procedure to the polynomials $1, x, x^2, \ldots$. Determine the first three Legendre polynomials.

(b) Determine the polynomial of 2nd-degree that best approximates the function e^x with respect to the inner product (1); that is, find the polynomial p that minimizes

$$\int_{-1}^{1} |p(x) - e^x|^2 \, dx$$

among all polynomials of degree at most 2.

3. (a) Reduce $u'' + 3u' + 2u = 0$ to a first order system and state it in matrix form.

(b) Find the eigenvalues and eigenvectors of the matrix.

(c) Find the general solution to the first order system.

(d) Use your result from (c) to find the general solution to the original second order equation.

(e) Any second order linear differential operator, including the above operator, can be converted into a self-adjoint operator. Do so.

4. (a) Find the two critical points for the first order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x^2 - x - y \end{pmatrix}$$

(b) Draw the phase portrait in neighborhoods of each of the two critical points.

VI. TOPOLOGY

1. Let $\{U_\alpha\}$ be a collection of open sets in the real line which covers a set A. Show that a countable subcollection of $\{U_\alpha\}$ covers A.

2. Show that the product $R \times [-1, 1]$ of the real numbers R and the interval [-1,1] with the dictionary ordering is connected. The dictionary ordering means $(x_1, y_1) < (x_2, y_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$.

3. Let S^1 be the circle of radius one in the plane and R be the set of real numbers. Show that if $f : S^1 \to R$ is a continuous function, then there are two distinct points $x, y \in S^1$ so that $f(x) = f(y)$.

4. Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to the function f, show f is continuous.