

WINTER 2001
APPLIED MATHEMATICS
PH.D. PRELIMINARY EXAMINATION
Option: Ordinary Differential Equations

Instructions: Give solutions to *exactly* 5 of the following 8 problems. (If you give more than 5 solutions, your grade will be based on the first 5 that appear.)

Some Notation:

| | |
|---------------------|---|
| \mathbb{N} | The set of natural numbers $\{1, 2, 3, \dots\}$. |
| \mathbb{R} | The set of real numbers. |
| $ \cdot $ | The standard Euclidean norm on \mathbb{R}^n . |
| \dot{u} | The first derivative of u with respect to time t . |
| $C(\Omega, \Omega)$ | The set of continuous functions from Ω to Ω . |

(1) Prove the following assertion or give an explicit counterexample:

There is no continuously differentiable function $x : (0, \infty) \rightarrow (0, \infty)$ satisfying $\dot{x}(t) \geq [x(t)]^2$ for every $t \in (0, \infty)$.

(2) Let \mathcal{K} be a nonempty compact metric space, let $\varphi : \mathbb{R} \times \mathcal{K} \rightarrow \mathcal{K}$ be a continuous dynamical system, and let p be a point in \mathcal{K} .

(a) Give a definition of $\omega(p)$, the omega-limit set of p (with respect to φ).

(b) Prove that $\omega(p)$ is invariant (with respect to φ) and connected.

(3) In items (i)–(iii), an autonomous differential equation in polar coordinates with phase space \mathbb{R}^2 is given, along with a particular solution $(\bar{r}, \bar{\theta})$ of the equation. For each of the items (i)–(iii), state which of the statements (a)–(c) is/are true:

(a) $(\bar{r}, \bar{\theta})$ is Lyapunov stable;

(b) $(\bar{r}, \bar{\theta})$ is asymptotically stable;

(c) $(\bar{r}, \bar{\theta})$ is orbitally stable.

(You do not need to justify your answers.)

(i)
$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = 1 \end{cases} ; (\bar{r}, \bar{\theta}) = (1, t)$$

(ii)
$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2 \end{cases} ; (\bar{r}, \bar{\theta}) = (1, t)$$

(iii)
$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases} ; (\bar{r}, \bar{\theta}) = (1, 0)$$

(4)

(a) State what it means for a continuous map to be *topologically transitive*.

(b) State what it means for a continuous map to be *topologically mixing*.

(c) Is it possible for a continuous map to be topologically transitive without being topologically mixing? If it is, give an explicit example. If it isn't, say so.

(d) Is it possible for a continuous map to be topologically mixing without being topologically transitive? If it is, give an explicit example. If it isn't, say so.

(5) Suppose that a and δ are real constants, that $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and that $k, x, X, y,$ and Y are in $C(\mathbb{R}, \mathbb{R})$. Assume that

$$\begin{aligned}x(t) &= x(a) + \int_a^t f(s, x(s)) ds + X(t), \\y(t) &= y(a) + \int_a^t f(s, y(s)) ds + Y(t),\end{aligned}$$

and

$$E(t) = |X(t)| + |Y(t)|$$

for every $t \in \mathbb{R}$, that $|f(t, u) - f(t, v)| \leq k(t)|u - v|$ for every $t, u, v \in \mathbb{R}$, and that $|x(a) - y(a)| \leq \delta$. Show that

$$|x(t) - y(t)| \leq E(t) + \delta \exp\left(\int_a^t k(s) ds\right) + \int_a^t E(s)k(s) \exp\left(\int_s^t k(\tau) d\tau\right) ds$$

for every $t \geq a$.

(6) Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be given by the formula $F(x) = 2x$, and let $G : \mathbb{R} \rightarrow \mathbb{R}$ be given by the formula $G(x) = ax$ for some constant a . Determine (with proof) the set of values of a for which F and G are topologically conjugate maps.

(7)

(a) State LaSalle's Invariance Principle.

(b) By considering the function $V(x, y) := y^2/2 + x^2 + x^3$, show that the solution of

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2x - ay - 3x^2\end{aligned}$$

at the origin is asymptotically stable for every $a > 0$.

(8) Consider the system

$$\begin{aligned}\dot{x} &= (b - 1)(-x + y) \\ \dot{y} &= \left(\frac{b - c - 1}{b - 1}\right)x - y - xz \\ \dot{z} &= z + xy,\end{aligned}$$

where b and c are real constants. Determine the dimension of $W_{\text{loc}}^u(0)$ (the local unstable manifold of the origin) as a function of b and c . (If you wish, you may specify a finite subset \mathcal{S} of \mathbb{R} and determine the dimension of $W_{\text{loc}}^u(0)$ only when $\{b, c\} \cap \mathcal{S}$ is empty.)