MASTER'S EXAMINATION IN MATHEMATICS

Saturday, 12 January 2002

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

I. Basic Analysis

II. Groups and Linear Algebra

III. Analysis

IV. Fields, Rings and Modules

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Prove that Cauchy sequences converge. You may use the fact that every bounded sequence has a convergent subsequence.

2. Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{x+1}{x-3} \right)^n .$$

3. Show that a non-singleton metric space with a countable number of points is not connected.

4. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ has the property that for some $0 \leq c < 1$

$$\|f(x) - f(y)\| \leq c\|x - y\| \quad \forall\ x, y \in \mathbb{R}^n$$

then $\exists\ x_0 \in \mathbb{R}^n$ with $f(x_0) = x_0$.

II. GROUPS AND LINEAR ALGEBRA

1. If G is a group, show that if H is a normal subgroup of G, then the multiplication $Hg_1Hg_2 = Hg_1g_2$ is well defined.

   Find an example of a group G with a subgroup H for which this multiplication would not be well defined.

2. Show that every finite group is isomorphic to a subgroup of S_n, the group of permutations of $\{1, 2, \dots, n\}$.

   Show that A_n, the group of even permutations is a normal subgroup of S_n.

3. Let A be an $m \times n$ matrix. Show that if $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$, then each solution of $A\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x}_0 + \boldsymbol{y}$ where $\boldsymbol{x}_0$ is fixed and $\boldsymbol{y}$ is a solution of $A\boldsymbol{y} = \mathbf{0}$.

   Show that if there is such a solution, then $\boldsymbol{b} \in$ column space of A.

4. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. If these are all distinct, prove that A is similar to a diagonal matrix. State clearly any result that you use.

III. ANALYSIS

1. Show that if $(X, \mathcal{M}, \mu)$ is a measure space and $1 \le p \le q \le r < \infty$, then

$$L^p(X, \mathcal{M}, \mu) \cap L^r(X, \mathcal{M}, \mu) \subseteq L^q(X, \mathcal{M}, \mu).$$

2. Let $\mathcal{L}$ be the collection of Lebesgue measurable subsets of $\mathbb{R}$, let λ be Lebesgue measure, and let $f \in L^1(\mathbb{R}, \mathcal{L}, \lambda)$ be given. Show that the function $\tau : \mathbb{R} \to L^1(\mathbb{R}, \mathcal{L}, \lambda)$ defined by $\tau(s) = f(\cdot + s)$ (*i.e.*, $\tau(s)(x) = f(x + s)$) is continuous.

3. State Lebesgue's Dominated Convergence Theorem, and prove it using Fatou's Lemma.

4. Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X.

   (a) Give a standard definition of what it means for a function $f : X \to \mathbb{R}$ to be $\mathcal{M}$-measurable.

   (b) Use the definition from part a to prove that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are $\mathcal{M}$-measurable then $f + g$ is $\mathcal{M}$-measurable

IV. FIELDS, RINGS AND MODULES

1. Does the polynomial $x^3 - x + 1$ have a root in the field with 81 elements? Justify your answer.

2. Let R be a commutative ring with 1, and let S be a subset of R with the following properties:

   (a) $1 \in S$.

   (b) 0 is not an element of S.

   (c) S is closed under multiplication.

   Prove that there exists a prime ideal of R which does not meet S. (Hint: Use Zorn's lemma.)

3. Let R be a commutative ring with 1, M and P finitely generated R-modules. Suppose we have a surjective R-module homomorphism $f : M \to P$.

   (a) If P is projective, show that the kernel of f is finitely generated.

   (b) Give an example to show that $\ker f$ need not be finitely generated if P is not projective.

4. Let R be a commutative ring with 1. Let M be a Noetherian R-module, and let $f : M \to M$ be a surjective R-module homomorphism. Prove that f is an isomorphism.

V. APPLIED MATHEMATICS

1. The system

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

describes the evolution of certain predator-prey populations. Assume a, b, c, d are the positive constants (respectively) $\frac{1}{2}, 1, 2, 1$. Find the steady-states of the system (equilibria) and discuss them–which are stable? Are solutions periodic nearby any of the equilibria? Observe appropriate theorems as needed.

2. The following equations describe heat transfer in a large slab of homogeneous material of thickness a where the faces of the slab are held at temperature T. Find the temperature function $u(x,t)$. Assume that $f(a) = f(0)$, k and T are constants. (State appropriate convergence theorems as needed.)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = T$$
$$u(a,t) = T$$
$$u(x,0) = f(x)$$

3. The following matrix A arises as the coefficient matrix of a system of ordinary differential equations, $\frac{dx}{dt} = Ax$. Find e^{At} by any method.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. (a) State Gauss's theorem (the divergence theorem) in standard vector notation.

   (b) State Stokes' theorem.

   (c) Verify Stokes' theorem for the velocity field $\boldsymbol{v} = xz\boldsymbol{j}$ for the surface $z = 4 - y^2$ cut off by the planes $y = x, x = 0, y = 0$.

VI. TOPOLOGY

1. Show that if X is a topological space with a countable basis, then every uncountable set $A \subset X$ has a limit point P which is an element of A.

2. Let X be a linearly ordered space with the linear topology. Show that X is regular. That is: for each $x \in X$ and each closed $A \subset X$ with $x \notin A$, show that there are disjoint open sets U and V such that $x \in U$ and $A \in V$.

3. Show that the product of regular spaces is regular.

4. If f is a continuous transformation of a compact metric space, A, into a metric space B, then f is uniformly continuous.