MASTER'S EXAMINATION IN MATHEMATICS

Saturday, 13 September 2003

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis.

I. Basic Analysis

II. Groups and Linear Algebra

III. Analysis

IV. Fields, Rings and Modules

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

Instructions. Do not assume so much that the problem becomes trivial.

1. Let $f : [a, b] \to \mathbb{R}$ be bounded. Suppose that for each n, Δ_n is a partition of $[a, b]$. Prove that if $\lim_{n\to\infty}[U(f, \Delta_n) - L(f, \Delta_n)] = 0$, then f is integrable and $\lim_{n\to\infty} U(f, \Delta_n) = \int_a^b f$.

2. State and prove the intermediate value theorem.

3. Prove that the uniform limit of continuous functions is continuous.

4. Given $f : \mathbb{R}^n \to \mathbb{R}^m$ 1 to 1 and smooth (the rank of the matrix Df is at least n), and $K \subseteq \mathbb{R}^n$ compact. Prove that for some $c > 0$, $\|f(u) - f(v)\| \geq c\|u - v\|$ $\forall\ u, v \in K$.

II. GROUPS AND LINEAR ALGEBRA

1. Let G be the additive group of polynomials in x with coefficients in $\mathbb{Z}$. Let H be the multiplicative group of all positive rationals. Prove that $G \cong H$.

2. Let G be any group, and $S = \{g^2 \mid g \in G\}$. Let H be a normal subgroup of G, with $S \subseteq H$. Prove that G/H is abelian.

3. Let M, N be non-singular integer n by n matrices with $\det(N) = \pm 1$, and such that $MN = D$ is diagonal. Let $\mathbb{Z}^n$ denote the additive group of integer n-tuples (1 by n matrices), and let $G = \{zM \mid z \in \mathbb{Z}^n\}$. Note that G is a subgroup of $\mathbb{Z}^n$. Prove that
$$\mathbb{Z}^n/G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_n}$$
where $d_1, d_2, \ldots, d_n$ are the diagonal entries in D.

4. A Fibonacci-type sequence is any sequence
$$(a_0, a_1, a_2, \ldots)$$
of real numbers such that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Show that the Fibonacci-type sequences form a vector-space of dimension two over the reals. You must define addition and scalar multiplication on the Fibonacci-type sequence.

1. Let $(\Omega, \mathcal{F})$ be a measure space with Ω a set and $\mathcal{F}$ a σ algebra of subsets and let $g : \Omega \to \mathbb{R}$. We say g is measurable if $g^{-1}((a,b)) \in \mathcal{F}$ for every interval (a,b). Show that this definition of measurability is equivalent to saying that $g^{-1}([a,\infty))$ is in $\mathcal{F}$ for every $a \in \mathbb{R}$.

2. Suppose f is a nondecreasing function defined on $[0,1]$ and $f'(x) = 0$ for Lebesgue a.e. points. Is it necessary that f' be Lebesgue measurable? Is it necessary that $\int_0^1 f'(x)\,dm = 0$? Explain.

3. Let $(\Omega, \mathcal{F})$ be a measure space with Ω a set and $\mathcal{F}$ a σ algebra of subsets and let $g_n : \Omega \to \mathbb{C}$ be a measurable function for each $n \in \mathbb{N}$. Also, for each $\omega \in \Omega$, $\{g_n(\omega)\}$ is a sequence of points in $\mathbb{C}$. Now let
$$C \equiv \{\omega : \{g_n(\omega)\} \text{ converges.}\}$$
Is it necessary that C be a measurable set? Prove or disprove.

4. Suppose $\{f_n\}$ is a sequence of complex valued functions in $L^p(\Omega)$ for $p > 1$ and $\mu(\Omega) < \infty$ which satisfy
$$\sup\left\{\int_\Omega |f_n(\omega)|^p\,d\mu\right\} < \infty$$
and $\lim_{n\to\infty} f_n(\omega) = 0$ for a.e. ω. Can it be concluded that $\lim_{n\to\infty} \int_\Omega |f_n(\omega)|\,d\mu = 0$? If so give a reason and if not, give a counter example.

5. Let $\mathbf{f} : U \to \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^n$. What does it mean for $\mathbf{f}$ to be differentiable at $\mathbf{x} \in U$? If $\mathbf{f}$ is differentiable at $\mathbf{x} \in U$, show that $\mathbf{f}$ is continuous at $\mathbf{x}$.

6. If $f(x) = \sum_{k=1}^\infty p_k(x)$ where the convergence is uniform on $[0,1]$ and each p_k is a polynomial, can it be concluded that $f'(x) = \sum_{k=1}^\infty p_k'(x)$? Explain.

7. Suppose K is a sequentially compact nonempty subset of $\mathbb{R}^n$ and suppose $f : K \to \mathbb{R}$ is lower semicontinuous. Give a proof that there exists $\mathbf{x} \in K$ such that $f(\mathbf{x}) \le f(\mathbf{y})$ for all $\mathbf{y} \in K$. If you don't know what a lower semicontinuous function is, assume f is continuous.

8. Consider the following conjecture: If S is an uncountable set of irrational numbers, it is necessary that S has a rational number as a limit point. Prove or disprove this conjecture.

9. Suppose $\{f_n\}$ is a sequence of decreasing functions defined on $(0,1]$ and that for each $x \in (0,1]$, it follows $\lim_{n\to\infty} f_n(x) = 0$. Is it necessarily the case that $\{f_n\}$ converges uniformly on $(0,1]$? Explain?

IV. FIELDS, RINGS AND MODULES

1. Let F be a field. Show that the ring $F[x]$ of polynomials in one variable over F is a principal ideal domain.

2. Prove that any finite subgroup of the multiplicative group of a field is cyclic.

3. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible quartic polynomial with $\mathrm{Gal}(f) = S_4$, and let α and β be two distinct roots of f in a splitting field over $\mathbb{Q}$. Determine the degree $[\mathbb{Q}(\alpha + \beta) : \mathbb{Q}]$.

4. Let R be a ring with 1. Prove that if M is a finitely generated R-module that is generated by n elements, then every quotient of M may be generated by n (or fewer) elements.

V. APPLIED MATHEMATICS

1. Solve the heat equation for the rod:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,\ t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t > 0,$$
$$u(x,0) = 3\sin \pi x - \sin 3\pi x, \qquad 0 < x < 1.$$

2. Find the LU–decomposition of the matrix

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}.$$

3. Find the general solution of the first order linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

4. Approximate $\cos 0.750$ using the Lagrange interpolating polynomial of degree 3 and the approximations

$$\cos 0.698 = 0.7661 \quad \cos 0.733 = 0.7432 \quad \cos 0.768 = 0.7193 \quad \cos 0.803 = 0.6946.$$

VI. TOPOLOGY

1. Show that the product of finitely many compact spaces is compact.

2. Give an example of a continuous bijective function which is not a homeomorphism.

3. Show that the 2-sphere is the one-point compactification of the Euclidean plane.

4. Show that a connected, locally path-connected space is path-connected.