MASTER'S EXAMINATION IN MATHEMATICS EDUCATION
Saturday, 18 January 2003

INSTRUCTIONS.

Answer a total of eight questions, with the restriction of exactly three questions from Mathematics Education and at most two questions from any other category.

I. Basic Analysis

II. Groups and Linear Algebra

III. Advanced Analysis

IV. Rings, Modules, Fields and Galois Theory

V. Applied Mathematics

VI. Topology

VII. Mathematics Education

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Given $f : [a, b] \to \mathbb{R}$, monotone. Provo that f is integrable.

2. Let $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}$ have continuous partial derivatives. If $K \subseteq U$ is compact and convex, prove that $f : K \to \mathbb{R}$ is Lipschitz.

3. Prove the intermediate value theorem.

4. State and prove the ratio test for series.

II. GROUPS AND LINEAR ALGEBRA

1. Prove that any two bases for a finite dimensional vector space V have the same number of elements.

2. Prove that a group of order 42 must have a normal subgroup of order 21.

3. Let G be a group, and $Z(G)$ be the center of G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

4. Let G be a group, and suppose that M and N are both normal subgroups of G. Suppose that $M \cap N = \{e\}$, where e is the identity of G. Prove that for any $m \in M$ and $n \in N$, $mn = nm$.

III. ANALYSIS

1. Let $(\Omega, \mathcal{F})$ be a measure space with Ω a set and $\mathcal{F}$ a σ algebra of subsets and let $g : \Omega \to \mathbb{R}$. We say g is measurable if $g^{-1}((a,b)) \in \mathcal{F}$ for every interval (a,b). Show that this definition of measurability is equivalent to saying that $g^{-1}([a,\infty))$ is in $\mathcal{F}$ for every $a \in \mathbb{R}$.

2. Let $\{E_i\}$ be a sequence of measurable sets with the property that
$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$
Let $S = \{\omega \in \Omega$ such that $\omega \in E_i$ for infinitely many values of $i\}$. Show $\mu(S) = 0$ and S is measurable.

3. Let $(\Omega, \mathcal{F})$ be a measure space with Ω a set and $\mathcal{F}$ a σ algebra of subsets and let $g_n : \Omega \to \mathbb{C}$ be a measurable function for each $n \in \mathbb{N}$. Also, for each $\omega \in \Omega, \{g_n(\omega)\}$ is a sequence of points in $\mathbb{C}$. Now let
$$C \equiv \{\omega : \{g_n(\omega)\} \text{ converges.}\}$$
Show that C is a measurable set.

4. Suppose $u_n(t)$ is a differentiable function for $t \in (a,b)$ and suppose that for $t \in (a,b)$,
$$|u_n(t)|,\ |u_n'(t)| < K_n$$
where $\sum_{n=1}^{\infty} K_n < \infty$. Show
$$(\sum_{n=1}^{\infty} u_n(t))' = \sum_{n=1}^{\infty} u_n'(t).$$
which shows that $(\sum_{n=1}^{\infty} u_n(t))'$ exists and equals $\sum_{n=1}^{\infty} u_n'(t)$.

5. Let $f(x) = x^2$ for $x \in [-\pi, \pi]$ and extend f to be 2π periodic and defined on the whole of $\mathbb{R}$. Find the Fourier series for f and use this or some other method to show $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{6}\pi^2$. Cite the necessary theorem which allows you to conclude the appropriate convergence.

6. Let $m(W) > 0$, W is measurable, $W \subseteq [a,b]$. Show there exists a nonmeasurable subset of W. **Hint**: Let $x \sim y$ if $x - y \in \mathbb{Q}$. Observe that $\sim$ is an equivalence relation on $\mathbb{R}$. Let $\mathcal{C}$ be the set of equivalence classes and let $\mathcal{D} \equiv \{C \cap W : C \in \mathcal{C}$ and $C \cap W \neq \emptyset\}$. Now use the axiom of choice to obtain an interesting set.

7. Consider the following conjecture: If S is an uncountable set of irrational numbers, it is necessary that S has a rational number as a limit point. Prove or disprove this conjecture.

IV. FIELDS, RINGS AND MODULES

1. Let K/L and L/F be algebraic extensions of fields. Prove that K/F is an algebraic extension of fields.

2. Let F be a field and $F[x]$ the polynomial ring over F.

   (a) Show that an ideal M of $F[x]$ is maximal if and only if $F[x]/M$ is a field.

   (b) Show that for $p(x) \in F[x]$ the quotient $F[x]/p(x))$ is a field if and only if $p(x)$ is irreducible.

3. Let p be an odd prime and let ζ_p be a primitive pth root of unity. Describe with proof the Galois group G of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$. What is the permutation action of G on the elements

$$\zeta_p + \zeta_p^{-1}, \zeta_p^2 + \zeta_p^{-1}, \ldots, \zeta_p^{(p-1)/2} + \zeta_p^{(1-p)/2}.$$

4. Let R be an integral domain and let M, N be finitely generated R-modules. Let

$$Tor(M) = \{m \in M| \text{ there is } r \in R, r \neq 0, \text{ with } rm = 0\}.$$

   Let $\phi : M \to N$ be an R-module isomorphism.

   (a) Show that $(Tor(M))$ is a submodule of M.

   (b) Show that $\phi(Tor(M)) = Tor(N)$.

   (c) If $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$, what is $Tor(M)$?

V. APPLIED MATHEMATICS

1. Solve the heat equation for $u = u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$$
$$u(x,0) = 1 - x^2/2.$$

with initial conditions $u(0,t) = 1$, $u(1,t) = 1$.

2. Determine a finite difference scheme for computing an approximation to the initial value problem

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}, \qquad x \in (0,1),\ t > 0$$
$$v(x,0) = f(x), \qquad x \in [0,1]$$
$$v(0,t) = v(1,t) = 0.$$

3. Find the Choleski factorization, $A = LL^t$, of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

4. Prove that the Cantor Middle Thirds set is uncountable. Determine the fractal dimension of this set.

VI. TOPOLOGY

1. If f is a continuous map from a topological space A into a topological space B, and C is a connected subset of A prove $f(C)$ is a connected subset of B.

2. Let X be a linearly ordered space with the linear topology. Show that X is regular. That is: for each $x \in X$ and each closed $A \subset X$ with $x \notin A$, show that there are disjoint open sets U and V such that $x \in U$ and $A \in V$.

3. A topological space X is normal if one-point sets are closed in X and that for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing Z and B, respectively. Is the product of normal spaces normal? Explain.

4. If every uncountable subset of a metric space S has a limit point, then S has a countable dense subset.

VII. MATHEMATICS EDUCATION

Write responses to any THREE (3) of the following five tasks.

1. Discuss qualitative research in mathematics education. Address:
   (a) Purposes
   (b) Methodologies
   (c) Theoretical frameworks
   (d) Use
   (e) Importance

2. Describe"Symbolic Interactionism" and discuss how this framework might impact a view of mathematical knowledge and learning. Give examples to support your assertions.

3. Characterize sat least two central issues in *mathematics education* which impact classrooms, students and teachers. How has the mathematics education research community addressed these issues? Provide examples where appropriate and support your assertions.

4. (a) Explain why $\frac{7}{5} \div \frac{2}{3} = \frac{7}{5} \times \frac{3}{2}$.
   (b) Now look at your solution. Discuss any representations you have used, and how they support your solution.
   (c) Think of how you might present this problem to a class to solve. At what age level? What difficulties might you anticipate? How might you design the instructional situation to increase the students' chances of success?

5. Do ONE of the following:
   (a) List the first 4 Van Hiele levels and, using quadrilaterals as a context, describe some questions and possible student responses that would help you assess a student's level of understanding of quadrilaterals.
   (b) Identify 4 roles of proof as described by DeVilliers and provide an example of each. Which of these roles would be the most effective in helping high school students understand proof and reasoning. Justify your claim.