1 Real analysis questions

1. Give an example of a measure space, $(\Omega, \mu, \mathcal{F})$, and a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f, such that inequality is obtained in Fatou's lemma.

2. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$ and $\mu(S) =$ number of elements in S. If
$$f : \Omega \to \mathbb{C}$$
what do we mean by $\int f d\mu$? Which functions are in $L^1(\Omega)$? Explain.

3. Suppose f is an increasing, possibly discontinuous function defined on an interval, $[a, b]$. Is it always the case that such a function is Riemann integrable? Explain your answer by either proving or disproving the assertion.

4. Let U be an open subset of $\mathbb{R}^n$ and let $\mathbf{f} : U \to \mathbb{R}^m$. Tell what it means for $\mathbf{f}$ to have a derivative at $\mathbf{x} \in U$ and prove or disprove the assertion that $\mathbf{f}$ is continuous at $\mathbf{x}$ whenever $\mathbf{f}$ is differentiable at $\mathbf{x}$.

5. Gronwall's inequality states that if $u(t) \leq u_0 + \int_0^t ku(s)\, ds$ where $k \geq 0$, then $u(t) \leq u_0 e^{kt}$. Prove this inequality.

6. Suppose $\mathbf{f} : K \to \mathbb{R}^m$ where K is a compact subset of $\mathbb{R}^n$. If $\mathbf{f}$ is continuous, does it follow that $\mathbf{f}$ is uniformly continuous? Prove your answer.

7. Let $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfies the Lipschitz condition,
$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$$
for all $t \in [0, T]$ and let $\mathbf{x}_0 \in \mathbb{R}^n$ be given. Show there exists a unique solution to the initial value problem,
$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0.$$
Hint: You might consider a norm on $C([0, T] : \mathbb{R}^n)$ of the form,
$$||\mathbf{x}|| \equiv \max\left\{e^{-\lambda t} |\mathbf{x}(t)| : t \in [0, T]\right\},$$
argue this is equivalent to the usual norm on this space and that by choosing λ appropriately, the map, $G : C([0, T] : \mathbb{R}^n) \to C([0, T] : \mathbb{R}^n)$ given by
$$G\mathbf{x}(t) \equiv \mathbf{x}_0 + \int_0^t \mathbf{f}(s, \mathbf{x}(s))\, ds$$
is a contraction map.

8. Suppose E_k is a measurable set and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Show that the set,
$$A \equiv \{\omega \in \Omega : \omega \in E_k \text{ infinitely often}\}$$
has measure zero and is a measurable set. **Hint:** Write the set of interest in terms of countable intersections and unions of the sets, E_k.

9. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let
$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$
Show S is measurable. **Hint:** Since these spaces are complete, it suffices to describe the set of all ω such that $\{f_k(\omega)\}$ is a Cauchy sequence. Let
$$A_{kl,n} \equiv \left\{\omega : |f_k(\omega) - f_l(\omega)| < \frac{1}{n}\right\},$$
a measurable set. Now write down the set which may be described as "for all n there exists m such that for all $k, l > m$, we have $\omega \in A_{kl,n}$ in terms of countable unions and intersections. This will be the set on which the above is a Cauchy sequence.

10. Show that if f is a function in $L^1(\Omega)$ for $(\Omega, \mu, \mathcal{S})$ a measure space, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$, then
$$\int_E |f|\, d\mu < \varepsilon.$$

11. Chebyshev's inequality says that if $f \in L^1(\Omega)$ and if $A_\delta \equiv \{x \in \Omega : |f(x)| \geq \delta\}$, then
$$\mu(A_\delta) \leq \frac{1}{\delta} \int_\Omega |f(x)|\, d\mu$$
Prove this inequality.

12. Show that if S is a nonempty bounded set of real numbers that $\sup(-S) = -\inf(S)$.

13. Suppose that $p, q, r > 0$ and that for $\theta \in [0,1]$,

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}.$$

Establish the following inequality.

$$\left(\int |f|^r \, d\mu\right)^{1/r} \leq \left(\int |f|^p \, d\mu\right)^{\frac{\theta}{p}} \left(\int |f|^q \, d\mu\right)^{\frac{1-\theta}{q}}.$$

14. Give an example of a measure space, $(\Omega, \mathcal{F}, \mu)$, and a sequence of functions, $\{f_n\}$ which converge uniformly to f on Ω and yet

$$\lim_{n\to\infty} \int f_n d\mu \neq \int f d\mu.$$

Can you give a simple condition which will suffice to say that uniform convergence implies convergence of the integrals? If so, give such a condition and if not, tell why such a condition does not exist.

15. Suppose $u_n(t)$ is a differentiable function for $t \in (a,b)$ and suppose that for $t \in (a,b)$,

$$|u_n(t)|, \ |u_n'(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$(\sum_{n=1}^{\infty} u_n(t))' = \sum_{n=1}^{\infty} u_n'(t).$$

16. If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** Consider the proof that any countable set in $\mathbb{R}$ has measure zero and consider the rational numbers, a countable set.

17. Give an example of sets $A_n \subseteq \mathbb{R}$ with $\cap_{n=1}^{\infty} A_n = \emptyset$, $A_n \supseteq A_{n+1}$, but $\lim_{n\to\infty} m(A_n) \neq 0$.

18. Given $1 > \varepsilon > 0$, show there exists an open set $E \subseteq [0,1]$ dense in $[0,1]$, and $m(E) = \varepsilon$. **Hint:** Recall the construction of the Cantor set.

19. Let $f \in L^1(\Omega)$ for $(\Omega, \mu, \mathcal{S})$ a measure space. Show that $\{x : f(x) \neq 0\}$ has σ finite measure. **Hint:**You need to show $\{x : f(x) \neq 0\}$ $= \cup_{j=1}^{\infty} F_j$, $\mu(F_j) < \infty$. Try letting $F_j = \left\{x : |f(x)| \geq \frac{1}{j}\right\}$.

20. Let $(\Omega, \mathcal{S}, \mu)$ be an arbitrary measure space and define $\bar{\mu} : \mathcal{P}(\Omega) \to [0,\infty]$ by

$$\bar{\mu}(S) = \inf\{\mu(E) : E \supseteq S \text{ and } E \in \mathcal{S}\}.$$

Show $\bar{\mu}$ is an outer measure. If $\overline{\mathcal{S}}$ is the set of $\bar{\mu}$ measurable sets in the sense of Caratheodory, show $\overline{\mathcal{S}} \supseteq \mathcal{S}$ and $\bar{\mu} = \mu$ on $\mathcal{S}$.

21. Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x+t)dt.$$

Note f is continuous at 0 and $f(0) > 0$. Remember continuity of translation in L^p.

22. Let $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$, let $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$. Show $f * g$ given by

$$f * g\,(x) \equiv \int f\,(x-y)\,g\,(y)\,dy$$

is uniformly continuous on $\mathbb{R}$ and $|(f*g)(x)| \leq ||f||_{L^p} ||g||_{L^{p'}}$. The measure is Lebesgue measure.

23. A set of functions, $\Phi \subseteq L^1$, is uniformly integrable if for all $\varepsilon > 0$ there exists a $\sigma > 0$ such that $\int_E |f\,|\,du < \varepsilon$ whenever $\mu(E) < \sigma$. Prove Vitali's Convergence theorem: Let $\{f_n\}$ be uniformly integrable, $\mu(\Omega) < \infty$, $f_n(x) \to f(x)$ a.e., and $|f(x)| < \infty$ a.e. Then $f \in L^1$ and $\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0$. **Hint:** You might try using Eggorov's theorem.

24. Suppose $f \in L^\infty \cap L^1$. Show $\lim_{p\to\infty} ||f||_{L^p} = ||f||_\infty$. **Hint:** You may use the fact that

$$\mu\left(\{x \in \Omega : |f\,(x)| \geq ||f||_\infty - \varepsilon\}\right) > 0$$

and that $|f\,(x)| \leq ||f||_\infty$ a.e. Also, if you find it easier, assume $(\Omega, \mu, \mathcal{S})$ is a finite measure space, $\mu\,(\Omega) < \infty$.

25. Suppose $\mu(\Omega) < \infty$. Show that if $1 \leq p < q$, then $L^q(\Omega) \subseteq L^p(\Omega)$.

26. Suppose $L \in \mathcal{L}(X,Y)$ where X and Y are two finite dimensional vector spaces and suppose L is one to one. Show there exists $r > 0$ such that for all $\mathbf{x} \in X$,

$$|L\mathbf{x}| \geq r\,|\mathbf{x}|\,.$$

Hint: Define $|\mathbf{x}|_1 \equiv |L\mathbf{x}|$, observe that $|\cdot|_1$ is a norm and then use the theorem proved earlier that all norms are equivalent in a finite dimensional normed linear space.

27. Let U be an open subset of X, $\mathbf{f} : U \to Y$ where X, Y are finite dimensional normed linear spaces and suppose $\mathbf{f} \in C^1(U)$ and $D\mathbf{f}(\mathbf{x}_0)$ is one to one. Then show $\mathbf{f}$ is one to one near $\mathbf{x}_0$. **Hint:** Show using the assumption that $\mathbf{f}$ is C^1 that there exists $\delta > 0$ such that if

$$\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}_0, \delta),$$

then

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}_1 - \mathbf{x}_2)| \leq \frac{r}{2} |\mathbf{x}_1 - \mathbf{x}_2|$$

then use Problem 26.

28. Suppose $M \in \mathcal{L}(X, Y)$ where X and Y are finite dimensional linear spaces and suppose M is onto. Show there exists $L \in \mathcal{L}(Y, X)$ such that

$$LM\mathbf{x} = P\mathbf{x}$$

where $P \in \mathcal{L}(X, X)$, and $P^2 = P$. **Hint:** Let $\{\mathbf{y}_1 \cdots \mathbf{y}_n\}$ be a basis of Y and let $M\mathbf{x}_i = \mathbf{y}_i$. Then define

$$L\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \text{ where } \mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i.$$

Show $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is a linearly independent set and show you can obtain $\{\mathbf{x}_1, \cdots, \mathbf{x}_n, \cdots, \mathbf{x}_m\}$, a basis for X in which $M\mathbf{x}_j = \mathbf{0}$ for $j > n$. Then let

$$P\mathbf{x} \equiv \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$$

where

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i.$$

29. Let $\mathbf{f} : U \to Y$, $\mathbf{f} \in C^1(U)$, and $D\mathbf{f}(\mathbf{x}_1)$ is onto. Show there exists $\delta, \epsilon > 0$ such that $\mathbf{f}(B(\mathbf{x}_1, \delta)) \supseteq B(\mathbf{f}(\mathbf{x}_1), \epsilon)$. **Hint:**Let

$$L \in \mathcal{L}(Y, X), \; LD\mathbf{f}(\mathbf{x}_1)\mathbf{x} = P\mathbf{x},$$

and let $X_1 \equiv PX$ where $P^2 = P$, $\mathbf{x}_1 \in X_1$, and let $U_1 \equiv X_1 \cap U$. Now apply the inverse function theorem to $\mathbf{f}$ restricted to X_1.

30. Let $\mathbf{f} : U \to Y$, $\mathbf{f}$ is C^1, and $D\mathbf{f}(\mathbf{x})$ is onto for each $\mathbf{x} \in U$. Then show $\mathbf{f}$ maps open subsets of U onto open sets in Y.

31. Suppose $U \subseteq \mathbb{R}^2$ is an open set and $\mathbf{f} : U \to \mathbb{R}^3$ is C^1. Suppose $D\mathbf{f}(s_0, t_0)$ has rank two and

$$\mathbf{f}(s_0, t_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

Show that for (s, t) near (s_0, t_0), the points $\mathbf{f}(s, t)$ may be realized in one of the following forms.

$$\{(x, y, \phi(x, y)) : (x, y) \text{ near } (x_0, y_0)\},$$

$$\{(\phi(y, z)\, y, z) : (y, z) \text{ near } (y_0, z_0)\},$$

or

$$\{(x, \phi(x, z), z,) : (x, z) \text{ near } (x_0, z_0)\}.$$

32. Suppose B is an open ball in X and $\mathbf{f} : B \to Y$ is differentiable. Suppose also there exists $L \in \mathcal{L}(X, Y)$ such that

$$||D\mathbf{f}(\mathbf{x}) - L|| < k$$

for all $\mathbf{x} \in B$. Show that if $\mathbf{x}_1, \mathbf{x}_2 \in B$,

$$||\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) - L(\mathbf{x}_1 - \mathbf{x}_2)|| \leq k\, ||\mathbf{x}_1 - \mathbf{x}_2||\,.$$

Hint: Consider

$$||\mathbf{f}(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)) - \mathbf{f}(\mathbf{x}_1) - tL(\mathbf{x}_2 - \mathbf{x}_1)||$$

and let

$$S \equiv \{t \in [0, 1] : ||\mathbf{f}(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)) - \mathbf{f}(\mathbf{x}_1) - tL(\mathbf{x}_2 - \mathbf{x}_1)|| \leq$$

$$(k + \epsilon)\, t\, ||\mathbf{x}_2 - \mathbf{x}_1||\}\,.$$

33. Let $\mathbf{f} : U \to Y$, $D\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x} \in U$, $B(\mathbf{x}_0, \delta) \subseteq U$, and there exists $L \in \mathcal{L}(X, Y)$, such that $L^{-1} \in \mathcal{L}(Y, X)$, and for all $\mathbf{x} \in B(\mathbf{x}_0, \delta)$

$$||D\mathbf{f}(\mathbf{x}) - L|| < \frac{r}{||L^{-1}||},\; r < 1.$$

Show that there exists $\epsilon > 0$ and an open subset of $B(\mathbf{x}_0, \delta)$, V, such that $\mathbf{f} : V \to B(\mathbf{f}(\mathbf{x}_0), \epsilon)$ is one to one and onto. Also $D\mathbf{f}^{-1}(\mathbf{y})$ exists for each $\mathbf{y} \in B(f(\mathbf{x}_0), \epsilon)$ and is given by the formula

$$D\mathbf{f}^{-1}(\mathbf{y}) = \left[D\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right]^{-1}.$$

Hint: Let

$$T_{\mathbf{y}}(\mathbf{x}) \equiv T(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x} - \mathbf{L}^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{y})$$

for $|\mathbf{y} - \mathbf{f}(\mathbf{x}_0)| < \frac{(1-r)\delta}{2||L^{-1}||}$, consider $\{T_{\mathbf{y}}^n(\mathbf{x}_0)\}$. This is a version of the inverse function theorem for $\mathbf{f}$ only differentiable, not C^1.

34. Denote by $C([0,T]:\mathbb{R}^n)$ the space of functions which are continuous having values in $\mathbb{R}^n$ and define a norm on this linear space as follows.

$$||\mathbf{f}||_\lambda \equiv \max\left\{|\mathbf{f}(t)| e^{\lambda t} : t \in [0,T]\right\}.$$

Show for each $\lambda \in \mathbb{R}$, this is a norm and that $C([0,T];\mathbb{R}^n)$ is a complete normed linear space with this norm.

35. Let $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose $\mathbf{f}$ satisfies a Lipschitz condition,

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$$

and let $\mathbf{x}_0 \in \mathbb{R}^n$. Show there exists a unique solution to the Cauchy problem,

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0,$$

for $t \in [0,T]$. **Hint:** Consider the map

$$G : C([0,T];\mathbb{R}^n) \to C([0,T];\mathbb{R}^n)$$

defined by

$$G\mathbf{x}(t) \equiv \mathbf{x}_0 + \int_0^t \mathbf{f}(s, \mathbf{x}(s))\, ds,$$

where the integral is defined componentwise. Show G is a contraction map for $||\cdot||_\lambda$ given in Problem 34 for a suitable choice of λ and that therefore, it has a unique fixed point in $C([0,T];\mathbb{R}^n)$. Next argue, using the fundamental theorem of calculus, that this fixed point is the unique solution to the Cauchy problem.

36. Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping which satisfies

$$d(T^n x, T^n y) \leq r d(x, y)$$

for some $r < 1$ whenever n is sufficiently large. Show T has a unique fixed point. Can you give another proof of Problem 35 using this result?

2 Complex variable questions

1. Suppose $\gamma : [a,b] \to \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t) - \gamma(s)| \leq K|s-t|$. Show γ is of bounded variation and that $V(\gamma, [a,b]) \leq K|b-a|$.

2. Let $\gamma : [a,b] \to \mathbb{C}$ be an arbitrary C^1 curve and let f, g have continuous derivatives on some open set containing $\gamma([a,b])$. Prove the usual integration by parts formula.

$$\int_\gamma f g' dz = f(\gamma(b)) g(\gamma(b)) - f(\gamma(a)) g(\gamma(a)) - \int_\gamma f' g dz.$$

3. Suppose f and $f' : U \to \mathbb{C}$ are analytic and $f(z) = u(x,y) + iv(x,y)$. Verify $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If u is a harmonic function defined on $B(0,r)$ show that $v(x,y) \equiv \int_0^y u_x(x,t)\, dt - \int_0^x u_y(t,0)\, dt$ is such that $u + iv$ is analytic.

4. Define a function $f(z) \equiv \overline{z} \equiv x - iy$ where $z = x + iy$. Is f analytic? Explain.

5. Show that if $u(x,y) + iv(x,y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall the Cauchy Riemann equations and

$$\nabla u(x,y) = \langle u_x(x,y), u_y(x,y)\rangle.$$

6. Show that $\sin(z+w) = \sin z \cos w + \cos z \sin w$.

7. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

8. If f is analytic, show that $z \to \overline{f(\overline{z})}$ is also analytic.

9. Find the integrals using the Cauchy integral formula.

   (a) $\int_\gamma \frac{\sin z}{z-i} dz$ where $\gamma(t) = 2e^{it} : t \in [0, 2\pi]$.

   (b) $\int_\gamma \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it} : t \in [0, 2\pi]$

   (c) $\int_\gamma \frac{\cos z}{z^2} dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$

   (d) $\int_\gamma \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 1 + \frac{1}{2}e^{it} : t \in [0, 2\pi]$ and $n = 0, 1, 2$.

10. Let $\gamma(t) = 4e^{it} : t \in [0, 2\pi]$ and find $\int_\gamma \frac{z^2+4}{z(z^2+1)} dz$.

11. Prove the binomial formula,

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

where

$$\binom{\alpha}{n} \equiv \frac{\alpha \cdots (\alpha - n + 1)}{n!}.$$

Can this be used to give a proof of the binomial formula, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$? Explain.

12. Let U be an open set and let f be analytic on U. Show that if $a \in U$, then $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ whenever $|z-a| < R$ where R is the distance between a and the nearest point where f fails to have a derivative. The number R, is called the radius of convergence and the power series is said to be expanded about a. You can use the Cauchy integral formula.

13. State and give a proof of the fundamental theorem of algebra.

14. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq 0$ on U. **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

15. Classify the singular points of the following functions according to whether they are poles or essential singularities. If poles, determine the order of the pole.

    (a) $\frac{\cos z}{z^2}$

    (b) $\frac{z^3+1}{z(z-1)}$

    (c) $\cos\left(\frac{1}{z}\right)$

16. Use contour integration to find the integral $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$.

17. Evaluate the integral $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx$.