Ph.D. Qualifying Exam in Analysis Feb. 2000

Work at least 7 problems from Real analysis and 3 from complex analysis. Clear explanation is required at all stages of each answer. You may quote standard theorems other than the one you may be trying to prove.

Real analysis questions

1. Let $\{E_i\}$ be a sequence of measurable sets with the property that

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega$ such that $\omega \in E_i$ for infinitely many values of $i\}$. Show $\mu(S) = 0$ and S is measurable.

2. Let f_n, f be measurable functions with values in $\mathbb{C}$. We say that f_n converges in measure if

$$\lim_{n\to\infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e. **Hint:** Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Use appropriate theorems to prove this.

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^1(\Omega)$. We say that $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathfrak{S}$

$$|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Show that $|\mathfrak{S}| \equiv \{|f| : f \in \mathfrak{S}\}$ is uniformly integrable if $\mathfrak{S}$ is. Also show that $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.

4. The theorem of de la Vallee Poussin states that if $\mathfrak{S}$ is a family of measurable functions defined on a finite measure space, (Ω, μ) and there exists a function, g, which is positive and increasing on $(0, \infty)$ with $\lim_{t\to\infty} g(t) = \infty$ and

$$\sup \left\{ \int_\Omega |f| \, g(|f|) \, d\mu : f \in \mathfrak{S} \right\} < \infty,$$

then $\mathfrak{S}$ is a uniformly integrable subset of $L^1(\Omega)$ having

$$\sup \left\{ \int |f(\omega)| \, d\mu : f \in \mathfrak{S} \right\} < \infty. \tag{0.1}$$

Prove this theorem.

5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^1(\Omega)$ has the property that whenever $\mu(E) > 0$,

$$\frac{1}{\mu(E)} |\int_E f d\mu| \leq C.$$

Show $|f(\omega)| \leq C$ a.e.

6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to [-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable.

7. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let

$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$

Show S is measurable.

8. Let $m(W) > 0,\ W$ is measurable, $W \subseteq [a, b]$. Show, using the axiom of choice, there exists a nonmeasurable subset of W. **Hint**: Let $x \sim y$ if $x - y \in \mathbb{Q}$. Observe that $\sim$ is an equivalence relation on $\mathbb{R}$. Let $\mathcal{C}$ be the set of equivalence classes. Let $\mathcal{D} = \{C \cap W : C \in \mathcal{C} \text{ and } C \cap W \neq \emptyset\}$. If $C \cap W \in \mathcal{D}$, pick exactly one element of $C \cap W$. Denote by A the collection of these points.

9. Show that if g is Borel measurable and f is measurable, then $g \circ f$ is also measurable. Give an example of a Lebesgue measurable function, f and a continuous function g such that $f \circ g$ is not Lebesgue measurable. **Hint:** The second part of this can be accomplished through the use of that strange function based on the Cantor set which is continuous, climbs from 0 to 1 and yet has its derivative equal to zero on the complement of the Cantor set. Call this function, h. Then consider $g_1(x) = h(x) + x$. Show g_1 maps the Cantor set into a set having positive measure and then use the theorem which says there is a nonmesurable subset of this set. Now consider $g \equiv g_1^{-1}$, a continuous function along with $f \equiv \mathcal{X}_S \circ g_1$.

10. If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** Consider the proof that a countable set of numbers has measure zero when applied to the rational numbers.

Complex analysis questions

1. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

2. If f is analytic, show that $z \to \overline{f(\overline{z})}$ is also analytic.

3. Let $f : U \to \mathbb{C}$ be analytic and $f(z) = u(x, y) + iv(x, y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

4. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

5. Suppose f is an entire function and that f has the property that whenever we write $f(z)$ as a power series expanded about a point w, it follows that at least one of the coefficients in the power series must equal zero. Show that f must be a polynomial. **Hint:** Define a set, A_n to be the points, w such that if $f(z) = \sum_{k=0}^{\infty} a_k (z - w)^k$, it follows $a_n = 0$. Thus A_n consists of the points where the power series of f centered at these points has the nth coefficient equal to zero. Argue that some A_n is uncountable and therefore has a limit point.

6. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq 0$ on U. State and prove a theorem about the uniqueness of the solutions to the equation, $u_{xx} + u_{yy} = 0$ in U and $u = f$ on ∂U. **Hint for the first part:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.