Phd Exam Fall 2002

Work at least 7 problems from the real analysis section and at least 3 from the complex analysis section.

Real Analysis

1. Give an example of a measure space, $(\Omega, \mu, \mathcal{F})$, and a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f, such that inequality is obtained in Fatou's lemma.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to [-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable. Note you can't add or subtract functions which have values in this space and expect the operations to be continuous.

3. Let E be a countable subset of $\mathbb{R}$. Show $m(E) = 0$.

4. Given $1 > \varepsilon > 0$, show there exists an open set $E \subseteq [0, 1]$ dense in $[0, 1]$, and $m(E) = \varepsilon$. **Hint:** Recall the construction of the Cantor set. Next show there exists a strictly increasing function, f, which has the property that its derivative equals zero on a set of positive measure.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) \equiv \left(1 + |\mathbf{x}|^2\right)^k$. Find the values of k for which f is in $L^1(\mathbb{R}^n)$. **Hint:** Use polar coordinates.

6. Let B be a Borel set in $\mathbb{R}^n$ and let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^n$. Suppose B has the following property. For each $\mathbf{x} \in \mathbb{R}^n$, $m(\{t : \mathbf{x} + t\mathbf{v} \in B\}) = 0$. Then show $m_n(B) = 0$. Note the condition on B says roughly that B is thin in one direction.

7. If $f : \mathbb{R}^n \to [0, \infty]$ is Lebesgue measurable, show there exists $g : \mathbb{R}^n \to [0, \infty]$ such that $g = f$ a.e. and g is Borel measurable.

8. Suppose E is a Lebesgue measurable set which has positive measure and let B be an arbitrary open ball and let D be a set dense in $\mathbb{R}^n$. Establish the result of Smítal, which says that under these conditions, $\overline{m_n}((E + D) \cap B) = m_n(B)$ where here $\overline{m_n}$ denotes the outer measure determined by m_n. Is this also true for X, an arbitrary possibly non measurable set replacing E in which $\overline{m_n}(X) > 0$? **Hint:** Let $\mathbf{x}$ be a point of density of E and let D' denote those elements of D, $\mathbf{d}$, such that $\mathbf{d} + \mathbf{x} \in B$. Thus D' is dense in B. Now use translation invariance of Lebesgue measure to verify there exists, $R > 0$ such that if $r < R$, we have the following holding for $\mathbf{d} \in D'$ and $r_{\mathbf{d}} < R$.

$$\overline{m_n}((E + D) \cap B(\mathbf{x} + \mathbf{d}, r_{\mathbf{d}})) \geq$$

$$m_n((E + \mathbf{d}) \cap B(\mathbf{x} + \mathbf{d}, r_{\mathbf{d}})) \geq (1 - \varepsilon) m_n(B(\mathbf{x} + \mathbf{d}, r_{\mathbf{d}})).$$

Argue the balls, $m_n(B(\mathbf{x} + \mathbf{d}, r_{\mathbf{d}}))$, form a Vitali cover of B.

9. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set

$$E - E = \{x - y : x \in E, y \in E\}.$$

Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x + t)dt.$$

Note f is continuous at 0 and $f(0) > 0$. Remember continuity of translation in L^p.

10. Suppose for all $f \in C_c(0,\infty)$, $||Af||_{L^p(0,\infty)} \leq K\,||f||_{L^p(0,\infty)}$ where A is a linear operator defined on $L^p(0,\infty)$. Does this inequality hold for all $f \in L^p(0,\infty)$? Explain why or why not. Here it is understood that the measure is ordinary Lebesgue measure.

11. Let f be in $L^1_{loc}(\mathbb{R}^n)$. Show Mf, the Maximal function, is Borel measurable. Recall

$$Mf(\mathbf{x}) \equiv \sup_{r>0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{x})|\, dx.$$

Complex Analysis

1. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

2. If f is analytic, show that $z \to \overline{f(\overline{z})}$ is also analytic.

3. Let $f : U \to \mathbb{C}$ be analytic and $f(z) = u(x,y) + iv(x,y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

4. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

5. Suppose f is an entire function and that f has the property that whenever we write $f(z)$ as a power series expanded about a point w, it follows that at least one of the coefficients in the power series must equal zero. Show that f must be a polynomial. **Hint:** Define a set, A_n to be the points, w such that if $f(z) = \sum_{k=0}^{\infty} a_k (z-w)^k$, it follows $a_n = 0$. Thus A_n consists of the points where the power series of f centered at these points has the n^{th} coefficient equal to zero. Argue that some A_n is uncountable and therefore has a limit point.

6. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq 0$ on U. State and prove a theorem about the uniqueness of the solutions to the equation, $u_{xx} + u_{yy} = 0$ in U and $u = f$ on ∂U. **Hint for the first part:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x,y) = \varepsilon x^2 + u(x,y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x,y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

7. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z) = z^n + a_{n-1}z^{n-1} \cdots + a_1 z + a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z) = -z^n$ and let γ be a large circle, $\gamma(t) = re^{it}$ for r sufficiently large.

8. Prove Liouville's theorem from the Cauchy integral formula.