Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2005

Instructions: Answer *exactly* 6 of the 10 questions given. If you do more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
9. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $|\lambda|$ – the total variation of a measure λ.
11. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
12. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
13. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
14. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
15. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
16. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform

1. Suppose that $f \in L^1(\mu)$. Prove that for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\int_E |f| \, d\mu < \epsilon$$

whenever $\mu(E) < \delta$.

2. Let $X = [0,1]$. Prove that if f is a complex Lebesgue measurable on X with $|f| \leq 1$, then there exists a sequence $\{g_n\}$ such that $g_n \in C([0,1])$, $|g_n| \leq 1$, and

$$f(x) = \lim_{n\to\infty} g_n(x) \text{ a.e.[m]}.$$

[You may assume Lusin's Theorem in your proof.]

3. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, prove that $L^p(\mu) \supset L^q(\mu)$.

4. Let H be a Hilbert space with inner product $(\cdot,\cdot)$. Prove that if L is a continuous linear functional on H, then there exists a unique $y \in H$ such that

$$Lx = (x, y) \text{ for all } x \in H.$$

5. Let $\{f_n\}$ be a sequence of continuous complex functions on a nonempty complete metric space X such that $f(x) = \lim f_n(x)$ exists for every $x \in X$ (i.e. $f_n \to f$ pointwise). Prove that there is a nonempty open set V and a number $M < \infty$ such that

$$|f_n(x)| < M \text{ for all } x \in V \text{ and for all } n = 1, 2, 3, \ldots.$$

6. Suppose that μ, λ, λ_1, and λ_2 are measures on a σ-algebra $\mathcal{M}$ with μ positive. Prove the following:
(a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$;
(b) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

7. Let μ be a complex Borel measure on $\mathbb{R}^k$. Define the symmetric derivative, $D\mu$, of μ with respect to m. Define a Lebesgue point of an $L^1(\mathbb{R}^k)$ function. Prove that if $\mu \ll m$ and f is the Radon-Nikodym derivative of μ with respect to m, then

$$D\mu = f \text{ a.e.}[m], \quad \text{and} \quad \mu(E) = \int_E (D\mu) \, dm.$$

[You may assume that almost every $x \in \mathbb{R}^k$ is a Lebesgue point of an $L^1(\mathbb{R}^k)$ function.]

8. If $f \in \text{Lip } 1$ on $[a, b]$, prove that f is absolutely continuous and that $f' \in L^\infty$.

9. Suppose that $(X, \mathcal{G}, \mu)$ and $(Y, \mathcal{H}, \lambda)$ are σ-finite measure spaces, and suppose that ψ is a measure on $\mathcal{G} \times \mathcal{H}$ such that

$$\psi(A \times B) = \mu(A)\lambda(B)$$

for all $A \in \mathcal{G}$ and all $B \in \mathcal{H}$. Prove that $\psi(E) = (\mu \times \lambda)(E)$ for all $E \in \mathcal{G} \times \mathcal{H}$. [You may assume that $\mathcal{G} \times \mathcal{H}$ is the smallest monotone class which contains all elementary sets.]

10. Prove that if f and g are $L^1(\mathbb{R})$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$. [You may assume that h is $L^1(\mathbb{R})$.]