Phd Exam Winter 2006

Work at 7 problems from the real analysis section and 3 from the complex analysis section.

Real Analysis

1. Let $\{\phi_k\}_{k=1}^{\infty}$ be a sequence of continuous functions such that $\int \phi_k(\mathbf{x})\,dm_n = 1, \phi_k(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, and ϕ_k equals zero for all $|\mathbf{x}| > 1/k$. Now suppose $f \in L^p(\mathbb{R}^n)$. Show $\lim_{k\to\infty} ||f * \phi_k - f||_{L^p(\mathbb{R}^n)} = 0$.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions mapping Ω to $\mathbb{R}$. Show the set on which this sequence converges must be measurable.

3. Suppose $\mathcal{F}$ is a uniformly integrable set of functions defined on a measure space, $(\Omega, \mathcal{F}, \mu)$. Let $|\mathcal{F}| \equiv \{|f| : f \in \mathcal{F}\}$. Show that $|\mathcal{F}|$ is also uniformly integrable.

4. Does there exist a strictly increasing function defined on $[0, 1]$ which has a derivative a.e. and this derivative equals zero on a set of positive measure? If there is, you need to describe one. If there isn't you need to explain why there isn't one.

5. The Maximal function, Mf for $f \in L^1_{loc}(\mathbb{R}^n)$ was defined as

$$Mf(\mathbf{x}) \equiv \sup_{r>0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} |f|\,dm_n$$

Show that Mf is Borel measurable. **Hint:** You might first show

$$\mathbf{x} \to f_r(\mathbf{x}) = \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} |f|\,dm_n$$

is continuous.

6. If $f : \mathbb{R}^n \to [0, \infty]$ is Lebesgue measurable, show there exists $g : \mathbb{R}^n \to [0, \infty]$ such that $g = f$ a.e. and g is Borel measurable.

7. It can be proved that for f an increasing continuous function defined on an interval, $[a, b]$, $f'(t)$ exists a.e. Does it follow that

$$f(x) - f(a) = \int_a^x f'(t)\,dt?$$

Prove by giving an argument why this is so or else disprove by citing a counter example.

8. Show that in any Hilbert space there exists a maximal orthonormal set.

9. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set

$$E - E = \{x - y : x \in E, y \in E\}.$$

Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x + t)dt.$$

Show f is continuous at 0.

10. Suppose $f \in L^2(\mathbb{R}^n)$ and that for all $\phi \in C_c(\mathbb{R}^n)$, the space of continuous functions with compact support,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\,dm_n = 0$$

Can it be concluded $f = 0$ a.e.? Either prove or disprove.

11. Suppose $\{f_n\}$ is a sequence in $L^2(\Omega)$ which converges weakly to $f \in L^2(\Omega)$. This means

$$(f_n, g)_{L^2} \to (f, g)_{L^2}$$

for all $g \in L^2(\Omega)$. Also suppose $f_n \to f$ pointwise. Can it be concluded that $f_n \to f$ in $L^1(\Omega)$? Here $(\Omega, \mathcal{F}, \mu)$ is a finite measure space, $\mu(\Omega) < \infty$. Either prove or give a counter example.

12. Does there exist an uncountable set of irrational numbers which has no rational number as a limit point? Explain.

13. Give an example of a function which is continuous at every irrational number but discontinuous at every rational number. Also explain why there cannot be a function which is continuous at every rational number and discontinuous at every irrational number. For the second part, you might want to show the points of continuity of a function are a G_δ set.

Complex Analysis

1. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

2. If f is analytic, show that $z \to \overline{f(\overline{z})}$ is also analytic.

3. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant.

4. Suppose f is an entire function and that f has the property that whenever we write $f(z)$ as a power series expanded about a point w, it follows that at least one of the coefficients in the power series must equal zero. Show that f must be a polynomial. **Hint:** Define a set, A_n to be the points, w such that if $f(z) = \sum_{k=0}^{\infty} a_k (z - w)^k$, it follows $a_n = 0$. Thus A_n consists of the points where the power series of f centered at these points has the n^{th} coefficient equal to zero. Next show some A_n has a limit point.

5. We say a real valued function, u is subharmonic if $u_{xx}+u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq m$ on U. **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small.

6. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Does this polynomial have any zeros, z such that $|z| > 2$? **Hint:** Use Rouche's theorem.

7. Evaluate $\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2} dx$.

8. Prove Liouville's theorem from the Cauchy integral formula.

9. Does there exist an entire function which maps $\mathbb{C}$ onto the upper half plane? Explain.