

Topology Qualifying Exam – 22 January 2005

Indicate very clearly which problems you would like to have graded. Your solutions should argue from fundamental principles and should not quote big theorems unless the problem explicitly suggests that you may do so.

Policy on misprints: If you decide that a problem has been incorrectly stated, explain your interpretation and solve that problem. You are not to reinterpret the problem so as to trivialize the problem.

Section I. Prove three of the following five theorems.

I.1. Prove that a compact Hausdorff space is both regular and normal.

I.2. Suppose that $A \times B \subset U \subset H \times K$, where A , B , H , and K are compact Hausdorff spaces and U is open in the product space $H \times K$. Show that there exist open subsets V of H and W of K such that $A \times B \subset V \times W \subset U$.

I.3. State and prove Urysohn's Lemma.

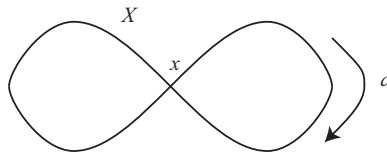
I.4. Suppose that H is a compact metric space, every point of which is a limit point of H . Prove that H is uncountable.

I.5. Let K be a countable subset of the plane \mathbf{R}^2 . Show that the complement $\mathbf{R}^2 \setminus K$ of K in the plane is arcwise connected.

Section II. Prove three of the following ten theorems.

II.1. Suppose that $p : (\tilde{U}, \tilde{u}) \rightarrow (U, u)$ is a covering projection, where the points \tilde{u} and u denote base points. Suppose further that V is locally and globally path connected. Suppose finally that $f : (V, v) \rightarrow (U, u)$ is a continuous function such that the images $f_*(\pi_1(V, v))$ and $p_*(\pi_1(\tilde{U}, \tilde{u}))$ of the fundamental groups induced by these maps satisfy the relationship that $f_*(\pi_1(V, v)) \subset p_*(\pi_1(\tilde{U}, \tilde{u}))$. Show that there is a map $\tilde{f} : (V, v) \rightarrow (\tilde{U}, \tilde{u})$ such that $p \circ \tilde{f} = f$. [You may assume the lifting property for paths and homotopies of paths.]

II.2. Consider the figure-eight X with base point x . (See the figure.) Let $G = \langle [a] \rangle$ be the subgroup of $\pi_1(X, x)$ generated by the loop a which circles the right hand loop in X exactly once. Draw an explicit picture of the covering space \tilde{X} whose projection $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ takes $\pi_1(\tilde{X}, \tilde{x})$ isomorphically onto G .



*Figure eight X , base point x ,
right-hand loop labelled a .*

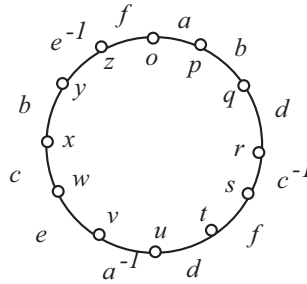
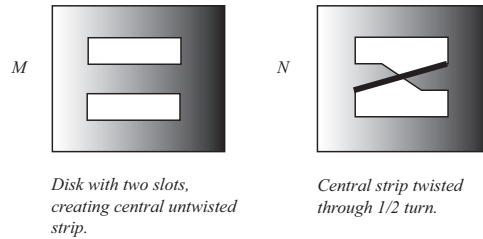
II.3. (a) Prove that the spaces M and N of the first figure at the top of page 2 are not homeomorphic. [You may use the fact that the boundary of a manifold is well-defined.]

(b) Prove that the spaces $M \times [0, 1]$ and $N \times [0, 1]$ are homeomorphic.

II.4. (a) Calculate the Euler characteristic of the 2-dimensional surface formed by the disk with edge identification $abdc^{-1}fda^{-1}ecbe^{-1}f$. (See the second figure on page 2.)

(b) Identify the surface obtained as a connected sum of spheres, tori, and/or projective planes.

II.5. Calculate the homology of the space $\mathbf{S}^1 \wedge \mathbf{S}^2 \wedge \mathbf{P}^2$, formed by identifying the base points of a circle \mathbf{S}^1 , a 2-sphere \mathbf{S}^2 , and a projective plane \mathbf{P}^2 . [You may assume that these spaces are simplicial complexes, and



Forming a 2-dimensional surface from a disk with indicated edge pairing.

*Vertices labelled o through z.
Edge identifications labelled a through f.*

you may apply the Mayer-Vietoris sequence.]

II.6. Suppose that $\cdots \rightarrow K_n \rightarrow K_{n-1} \rightarrow \cdots \rightarrow K_0 \rightarrow 0 \cdots$ is a free chain complex, that $\cdots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_0 \rightarrow 0 \cdots$ is an acyclic chain complex, and that $\phi_0 : K_0 \rightarrow L_0$ is a homomorphism. Prove that there is a chain map $\phi : K \rightarrow L$ which equals ϕ_0 in dimension 0.

II.7. Calculate the relative homology groups of the pair $(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$, where \mathbf{R}^n denotes Euclidean n -dimensional space. [You may assume the axioms of homology and the Mayer-Vietoris sequence.]

II.8. Use homology duality to prove that, if M^n is a connected, orientable manifold of odd dimension n , then the Euler characteristic $\chi(M^n)$ is 0.

II.9. State the differentiable inverse mapping theorem and the differentiable implicit function theorem. Use the former to prove the latter.

II.10. Consider the interior U of the disk of radius 1 in the plane, centered at the origin; and consider the open upper half plane $H = \{(x, y) \mid y > 0\}$. Endow U with the Riemannian metric $4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$ and H with the Riemannian metric $(dx^2 + dy^2)/y^2$.

- (a) Show that there is a linear fractional transformation taking U to H .
- (b) Show that this transformation is an isometry from the metric on U to the metric on H .