FALL 2002
APPLIED MATHEMATICS
PH.D. PRELIMINARY EXAMINATION

Option: Ordinary Differential Equations

Instructions: Give solutions to exactly 5 of the following 8 problems. If you give more than 5 solutions, your grade will be determined by the first five that appear.

Some Notation

$\mathbb{R}$ – the set of real numbers

$\dot{x}$ – the derivative of x with respect to t

ϕ_t – the flow generated by $\dot{x} = f(x)$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$

$\sum_2$ – the set of bi-infinite sequences of elements of $\{0, 1\}$

σ – the shift map on $\sum_2$

1. (a) State the definition of a section for ϕ_t.

(b) State the definition of a Poincaré section.

(c) State the definition of a Poincaré map.

(d) Given a section S for ϕ, and a $p \in S$ and smallest positive $T > 0$ for which $\phi_T(p) = p$, prove that there is an open subset $\sum$ of S containing p on which the Poincaré map is smooth.

2. (a) State the definitions of a Lyapunov function and a strict Lyapunov function for $\dot{x} = f(x)$ at x_0.
(b) State Lyapunov's Stability Theorem.
(c) Using Lyapunov's Stability Theorem, determine the stability of the rest points of

$$\begin{aligned} \dot{x}_1 &= x_2 - (x_1)^3 \\ \dot{x}_2 &= -x_1 - (x_2)^3 \end{aligned}$$

3. Suppose that ϕ is a complete flow.

(a) State the definition of the omega limit set, $\omega(p)$, of the orbit $t \to \phi_t(p)$ through p.

(b) Prove that if the forward orbit of ϕ through p, which is the set $\{\phi_t(p) : t \geq 0\}$, has compact closure, then $\omega(p)$ is nonempty, compact, and connected.

(c) Give an example to show that the compact closure hypothesis is necessary.

4. (a) State the definition of a fundamental matrix solution of $\dot{x} = A(t)x$.

(b) Assuming that $t \to \Phi(t)$ is a fundamental matrix solution of $\dot{x} = A(t)x$ defined on an interval J_0 of t_0, derive the Variation of Constants Formula for the solution of the initial value problem

$$\dot{x} = A(t)x + g(x, t), \quad x(t_0) = x_0.$$

5. (a) State Floquet's Theorem.

(b) State the definition of a characteristic multiplier for a T-periodic system $\dot{x} = A(t)x$.

(c) Assuming the Variation of Constants Formula and Floquet's Theorem, prove that if 1 is not a characteristic multiplier of the T-periodic system $\dot{x} = A(t)x$, then $\dot{x} = A(t)x + b(t)$, where $b(t)$ is T-periodic, has a least one T-periodic solution.

6. (a) State the definition of a hyperbolic rest point.

(b) State the Hartman-Grobman Theorem for a hyperbolic rest point of a flow.

(c) Using the Hartman-Grobman Theorem, sketch the phase portrait in a neighbourhood of each hyperbolic rest point of

$$\begin{aligned} \dot{x}_1 &= 2x_2 - (x_1)^2 \\ \dot{x}_2 &= -x_1 + x_2. \end{aligned}$$

7. (a) Given a homeomorphism ψ between the Smale Horseshoe and $\sum_2$, prove that the Smale Horseshoe has countably many periodic orbits, uncountably many non-periodic orbits, and a dense orbit.

(b) Explain what it means mathematically for the dynamics on the Smale Horseshoe to be chaotic.

8. (a) State what it means for a fixed point of a diffeomorphism f on a manifold M to be hyperbolic.

(b) State the Stable Manifold Theorem for a Hyperbolic Fixed Point of a Diffeomorphism.

(c) Determine the dimensions of the local stable manifolds of the hyperbolic fixed points of the diffeomorphism

$$f(x_1, x_2) = (0.5x_1 - x_2, 0.5(x_1)^3 + 2x_2).$$