Get Knotted

1 Knots

Informally a *knot* is a knotted loop of string. You can create one by knotting an extension chord and then joining the ends, or in drawing a diagram such as those below. These are called *knot projections*.

Figure 1: Trefoil Knot

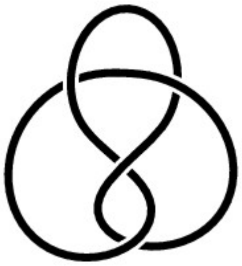


Figure 2: Figure 8 Knot

A *link* is a knot which has several loops of string. Some famous links are 2_1^2 the Hopf link, 5_1^2 the Whitehead link, and 6_2^3 the Borromean rings.

A *crossing* is a place where the knot crosses itself. The first number in a knot's "name" is the number of crossings. Can you figure out what the other numbers mean?

Some common questions in knot theory deal with the following:

1. How can we tell knots apart?

2. Is the figure 8 knot really the unknot? Or is the figure 8 knot really the trefoil knot?
3. Can we pull apart the Borromean rings without breaking them?

2 Getting Knotted

Activity

3 Reidemeister moves

There are many ways of representing the same knot. For example, below are two representations of the unknot.

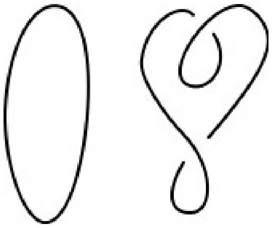
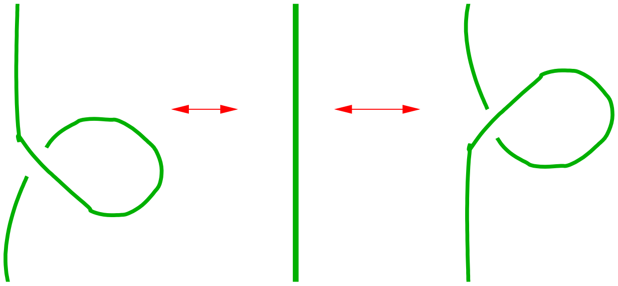


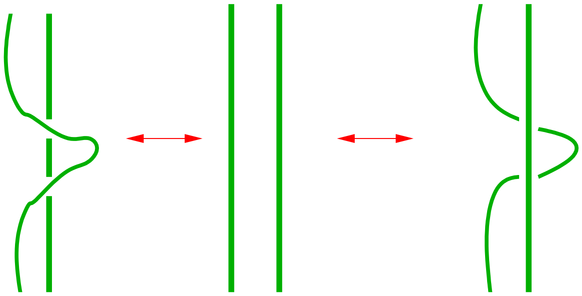
Figure 3: Two representations of the unknot

Actually any of the following moves on the next page do not change the knot. In 1920 Kurt Reidemeister proved the following theorem.

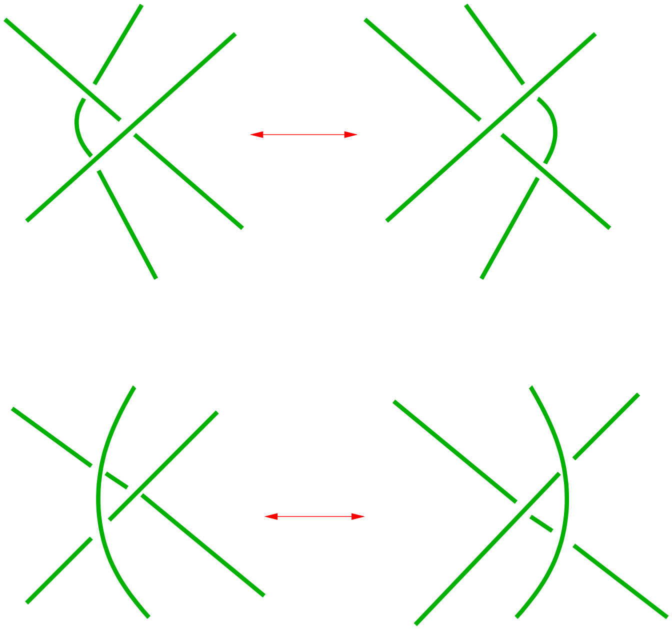
Theorem 1 *If two knot diagrams represent the same knot (or a link) then one can be transformed to the other by a sequence of Reidemeister moves.*



Type I Reidemeister move



Type II Reidemeister move



Type III Reidemeister move

Exercises

1. Start with the knot diagram for the trefoil knot and change one of the crossings (i.e. make the upper strand go under the other one). Show that this new knot is an unknot.

2. Do the same for the figure 8 knot.

3. The *mirror image* of a knot is obtained by reversing all crossings. Show that the mirror image of the figure 8 knot is the same knot as the figure 8 knot. How many moves did you use?

4. Is the mirror image of the trefoil knot the trefoil knot?

4 Tricolorability

A *strand* in a knot diagram is a continuous piece that goes from one undercrossing to the next. Note: The number of strands in a knot is the same as the number of crossings.

A knot or a link is *tricolorable* if each strand can be colored in one of three colors satisfying:

1. At least two colors are used, and
2. at each crossing either all three colors are present or only one color is present.

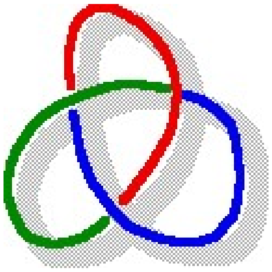


Figure 4: The Trefoil knot is tricolorable

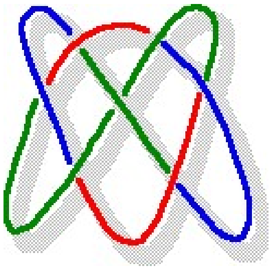


Figure 5: The 7_4 knot is tricolorable

Exercises

1. Decide which of the following are tricolorable:

(a) the unknot

(b) the figure 8 knot

(c) the 2-component unlink

(d) the Hopf link

(e) the Whitehead link

(f) the 3-component unlink

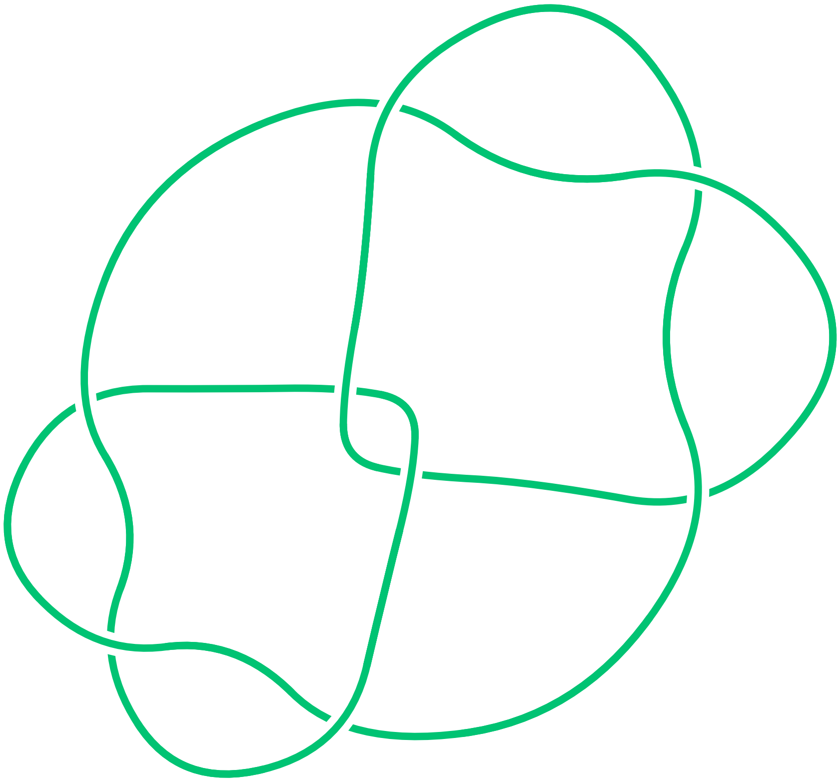
(g) the Borromean ring

(h) Which knots with 5, 6, and 7 crossings?

2. Show that if you start with a tricolorable knot diagram and you perform a Reidemeister move, the new knot diagram is also tricolorable. Conclude the following:
   Some knots are tricolorable and some are not, but to find out it is enough to check a single knot diagram for this knot.

3. Show that the trefoil is really a knot. Also show that the figure 8 knot is different from the trefoil, that the Hopf and Whitehead links cannot be pulled apart, and that Borromean rings cannot be pulled apart.

4. Show that the "true lover's knot" is tricolorable.



True lover's knot

5 Linking Number

Now let's think about 2-component links. We will color one component red and the other blue. We will also choose a sense of traversing each string or an *orientation* of the string.

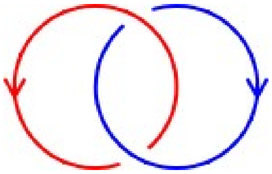


Figure 6: Hopf Link has linking number -1

We want to measure how many times one component "wraps around" the other. This is called the *linking number* and can be computed as follows. Look for those crossings where the red string is above the blue string. To each such crossing assign either a $+1$ or a -1 according to the right-hand rule.

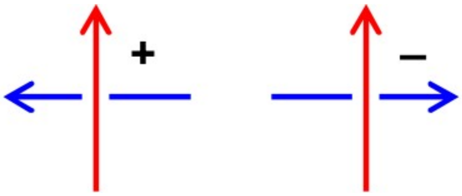


Figure 7: The right-hand rule for the linking number

Exercises

1. Compute the linking number for the unlink of 2 components, for the Hopf link, and for the Whitehead link.

2. What happens to the linking number if we reverse the orientation of one of the components?

3. What happens to the linking number if we perform a Reidemeister move?

4. Conclude that the Whitehead and Hopf links are really different.

5. What happens to the linking number if we switch the colors?

6. Examine the list of 2-component links at the end of the handout. Using the linking number and tricolorability, how many can you tell apart?

7. Use linking numbers to show that 7^3_1 and 8^3_2 are different links. Find other pairs of 3-component links that you can tell apart.

6 The Jones' Polynomial

In the early 90's a mathematician at UC Berkeley named Vaughn Jones discovered a new way to tell knots apart. For this work he received the Fields Medal, the highest award in mathematics. He assigns a polynomial to every knot. If the polynomials are different, the knots are also different. It is possible for different knots to have the same Jones' polynomial, but it happens rarely. This polynomial is a remarkably good method of distinguishing knots. We will go through the construction of this polynomial that is due to Louis Kauffman.

The first step is to assign a *bracket* $< K >$ to every knot or link diagram K. This is going to be a polynomial, initially in variables A, B, C, and it will satisfy the following rules:

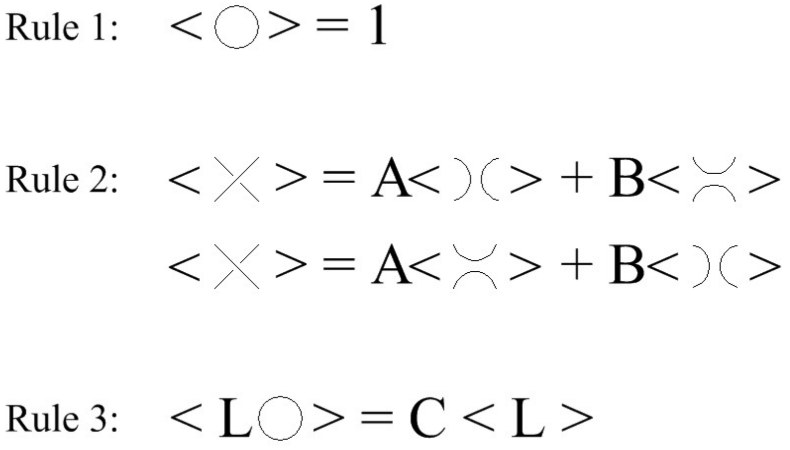


Figure 8: Bracket Rules

Now we want to make sure that things don't change when we perform a Reidemeister move.

1. To ensure that the bracket does not change under the Reidemeister move

of type II show that we need the following relations between A, B, C:

$$A^2 + ABC + B^2 = 0, BA = 1$$

Then what should B and C be?

2. Check that the bracket does not change after a Type III Reidemeister move.

3. Compute the bracket of the unlink with 2 components and with 3 components.

4. Compute the bracket of the Hopf link.

We still need to check the Type I Reidemeister move. This gives us a problem. The most interesting resolution of this problem involves the notion of the *writhe* of a knot projection.

7 Writhe

This is very similar to the concept of the linking number except that we are "color-blind." Orient all strings in the diagram and count with signs according to the right-hand rule.

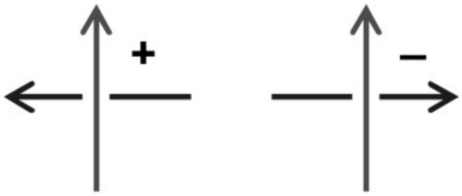


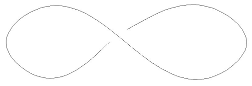
Figure 9: The writhe

1. What is the writhe of the standard picture of the trefoil? What is the writhe of the unknot?

2. How does the writhe change if we reverse all orientations?

3. How does the writhe change under the Reidemeister moves?

4. Show that no matter how hard you try you will not be able to transform the following into the circle with no crossings without using at least one Reidemeister move of Type I.



5. Show that $X(L) = (-A^3)^{-w(L)} < L >$ is unchanged under all Reidemeister moves. Here L is a knot projection and $w(L)$ is the writhe of L.

6. Compute $X(L)$ for the trefoil.

7. Compute $X(L)$ for the figure 8 knot. Deduce that the figure 8 knot is really a knot!

8. What happens to $X(L)$ if we reverse all crossings? Deduce that the trefoil is not the same as its mirror image. We say that the trefoil is *chiral*. Recall that the figure 8 knot is the same as its mirror image. It is *achiral*.

The polynomial $X(L)$ is essentially the same as the original Jones' polynomial (replacing A by $t^{-1/4}$ would give the original version). You have seen that it is hard to compute it, particularly if there are many crossings. But computers can handle this computation without much trouble. There is a program called *knotscape* that performs this task.

8 Resources

These notes were based off of Mladen Bestvina's Math Circle notes on Knots found at http://www.math.utah.edu/mathcircle/notes/knots.pdf

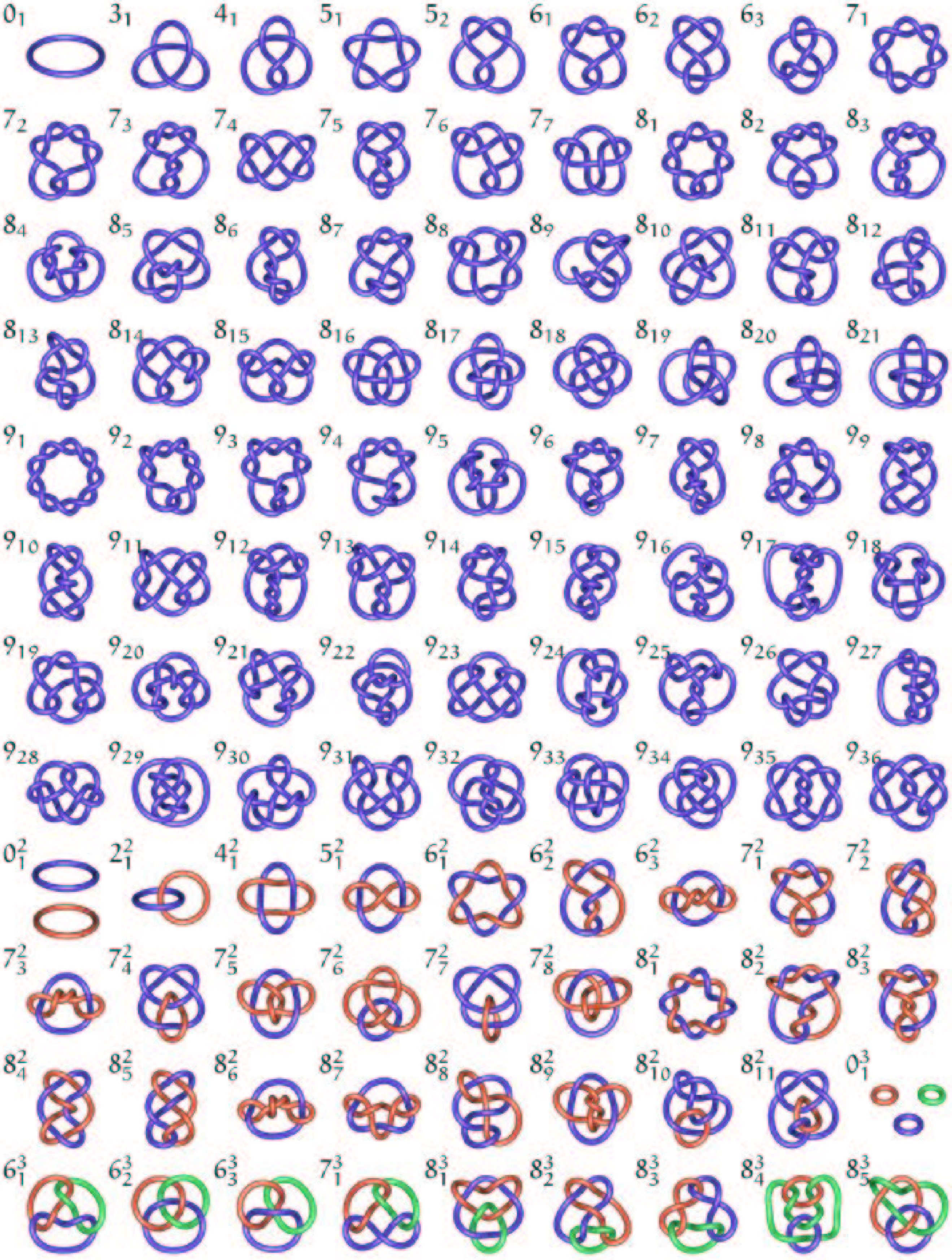
Other good resources are:

- Colin C. Adams: *The Knot Book*, W.H. Freeman and Company, New York, 1999
- http://www.knotplot.com/

This is Robert Scharein's excellent web site, packed with cool pictures, movies, and further links. He is the author of *knotplot*, software that produces such pictures.

- http://www.math.utk.edu/ morwen/knotscape.html

Morwen Thistlethwaite's program that computes various knot polynomials. You draw a knot with the mouse and it computes the polynomials.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