Name ______________________ Instructor______________________Section No. ________

Student Number ______________________

Math 112 – Fall 2005

Departmental Final Exam

Instructions:

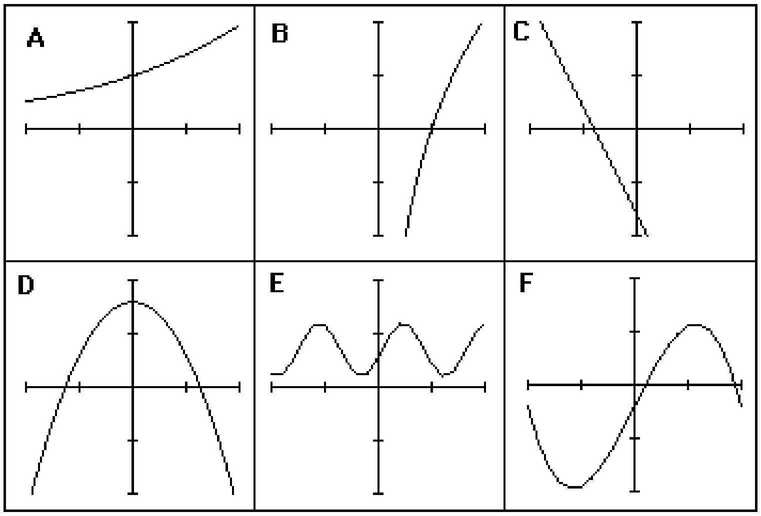
- The time limit is 3 hours.
- Problem 1 is a matching question; each correct match is worth 1 point.
- Problems 2(a) through 2(g) are true-false questions, each worth 1 point.
- Problems 3 through 10 are multiple choice questions, each worth 3 points.
- For problems 11 through 22, give the best answer and *justify* it by giving suitable reasons and/or by showing relevant work.
- Work on scratch paper will not be graded.
- Please write neatly.
- Notes, books, and calculators are not allowed.
- Expressions such as $\ln(1)$, e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.

For administrative use only:

1	/6
2	/7
M.C.	/24
11	/4
12	/4
13	/8
14	/5
15	/4
16	/4
17	/3
18	/6
19	/4
20	/6
21	/6
22	/9
Total	/100

Matching.

1. (6 points) Assign each graph to its *most likely* family of functions. (Some families are not represented—leave them blank.)



__F__ Cubic __A__ Exponential _____ Hyperbolic

__C__ Linear __B__ Logarithmic _____ Logistic

__D__ Quadratic __E__ Sinusoidal _____ Surge

True or False.

2. (7 points) Mark T if true under all circumstances; mark F otherwise.

__T__ a) The function $f(x) = x^2$ does not change concavity.

__T__ b) A cubic function on $\mathbb{R}$ always has an inflection point.

__T__ c) The function $f(x) = \frac{1}{x}$ changes concavity but has no inflection point.

__T__ d) A strictly increasing function defined on $\mathbb{R}$ may have a critical point.

__F__ e) A strictly increasing function defined on $\mathbb{R}$ may have an extremum.

__T__ f) A global extremum is also a local extremum.

__T__ g) A continuous function may not be differentiable.

Multiple Choice. Use the following chart to show your choices for Problems 3–10. Shade in your choice. Only this chart will be graded on these problems.

Problem 3	(a)	(b)	(c)	■	(e)	(f)
Problem 4	(a)	(b)	(c)	■	(e)	(f)
Problem 5	■	(b)	(c)	(d)	(e)	(f)
Problem 6	(a)	■	(c)	(d)	(e)	(f)
Problem 7	(a)	■	(c)	(d)	(e)	(f)
Problem 8	(a)	(b)	(c)	(d)	■	(f)
Problem 9	(a)	(b)	(c)	(d)	■	(f)
Problem 10	(a)	■	(c)	(d)	(e)	(f)

3. (3 points) $\int_0^3 2x\cos(x^2+1)\,dx =$

(a) $\int_0^9 \cos u\,du$ (b) $\left(\int_0^3 2x\,dx\right)\left(\int_0^3 \cos(x^2+1)\,dx\right)$ (c) $\int_1^{10} \cos(u+1)\,du$

(d) $\int_1^{10} \cos u\,du$ (e) $\int_0^{10} \cos 2x\,dx$ (f) None of the above.

4. (3 points) The integral $\int_{-2}^2 (x^4+3x^3-7x^2+5x-3)\,dx$ may be simplified to

(a) $2\int_0^2 (x^4-7x^2)\,dx$ (b) $\int_{-2}^2 (2x^4-7x^2-3)\,dx$ (c) $2\int_0^2 (3x^3+5x)\,dx$

(d) $2\int_0^2 (x^4-7x^2-3)\,dx$ (e) $2\int_0^2 (3x^3+5x-3)\,dx$ (f) None of the above.

5. (3 points) A particle in a force field experiences an acceleration $a(t) = 15\sin 3t$. If the initial position is $s(0) = 0$ and the initial velocity is $v(0) = 6$, find the position function $s(t)$.

(a) $11t - \frac{5}{3}\sin 3t$ (b) $11t - \frac{5}{3}\sin 3t + \frac{5}{3}$ (c) $t - \frac{5}{3}\sin 3t$

(d) $t - \frac{5}{3}\sin 3t - \frac{2}{3}$ (e) $11 - 5\cos 3t$ (f) None of the above.

6. (3 points) The position (in centimeters) of a particle at time t (in seconds) is given by $s(t) = t^3 + t + 8, t \in [0,1]$. Find a time $t_1 \in (0,1)$ such that the velocity of the particle at time t_1 is equal to its average velocity over the time interval $[0,1]$.

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$

(d) ± 1 (e) No such t_1 exists. (f) None of the above.

7. (3 points) The base of a rectangle is increasing at the steady rate of 3 cm/sec at the same time that the altitude of the rectangle is decreasing at the rate of 0.5 cm/sec. At a particular instant, the rectangle is a square with side 20 cm. At that instant, how is its area changing?

(a) 40 cm^2/sec (b) 50 cm^2/sec (c) 60 cm^2/sec

(d) 70 cm^2/sec (e) The area is not changing. (f) None of the above.

8. (3 points) Given $f(x) = x^2 + 1$ and $g(x) = \ln x$, compute $\frac{d}{dx}[g(f(x))]$.

(a) $2x + \frac{1}{x}$ (b) $\frac{2\ln x}{x}$ (c) 2

(d) $2x \ln x + \frac{x^2+1}{x}$ (e) $\frac{2x}{x^2+1}$ (f) None of the above.

9. (3 points) Given the position function $s(t) = t - \cos 2t, t \geq 0$, what is the velocity when the acceleration first becomes zero?

(a) $-2\sqrt{3}$ (b) -1 (c) 1 (d) 2 (e) 3 (f) None of the above.

10. (3 points) Walter wants to estimate $\sqrt{15}$ and decides to use Newton's method to find the positive zero of $f(x) = x^2 - 15$. If he chooses $x_1 = 4$, what does he get for x_2?

(a) 3.873 (b) 3.875 (c) 4 (d) 4.125 (e) x_2 is undefined. (f) None of the above.

Essay Problems. Write out your solutions to Problems 11-22 in the space provided. Show enough work to reveal your thought processes. Make the obvious simplifications.

11. (4 points) Given $f(x) = \dfrac{x+1}{x-2}$, find a formula for $f^{-1}(x)$.

SOLUTION:

$$x = \frac{y+1}{y-2}$$

Switching variables: 2pts. $\quad xy - 2x = y + 1$

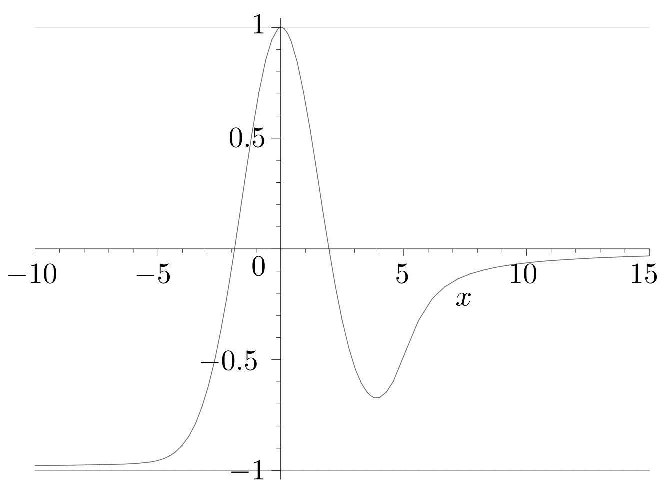
Correctly solving: 2pts. $\quad xy - y = 2x + 1$

-1 per mistake $\quad (x-1)y = 2x + 1$

$$f^{-1}(x) = y = \frac{2x+1}{x-1}$$

12. (4 points) Sketch the graph of a continuous function f defined on $(-\infty, \infty)$ and satisfying the properties

$$-1 \le f(x) \le 1, \quad f(0) = 1, \quad \lim_{x\to\infty} f(x) = 0, \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -1.$$



each condition is 1 pt.

not continuous: -1

13. (8 points) Find the limits. If the limit is ∞ or $-\infty$, so state; if the limit does not exist but is not ∞ or $-\infty$, write DNE.

(a) $\displaystyle\lim_{x\to3} \frac{x^2-9}{x-3}$ $\boxed{= \lim_{x\to3}(x+3) = 6}$

(b) $\displaystyle\lim_{x\to0} \frac{e^x-1}{x}$ $\boxed{= \lim_{x\to0} \frac{e^x}{1} = 1}$

(c) $\displaystyle\lim_{x\to0} \frac{2^x-1}{x}$ $\boxed{= \lim_{x\to0} \frac{2^x \ln 2}{1} = \ln 2}$

(d) $\displaystyle\lim_{x\to4^+} \frac{|x-4|}{x-4}$ $\boxed{= \lim_{x\to4^+} 1 = 1}$

(e) $\displaystyle\lim_{x\to4^-} \frac{|x-4|}{x-4}$ $\boxed{= \lim_{x\to4^-} (-1) = -1}$

(f) $\displaystyle\lim_{x\to4} \frac{|x-4|}{x-4}$ $\boxed{\text{DNE}}$

(g) $\displaystyle\lim_{x\to\infty} \frac{1}{x-3}$ $\boxed{= 0}$

(h) $\displaystyle\lim_{x\to3^+} \frac{1}{x-3}$ $\boxed{= \infty}$

14. (5 points) (a) State the definition of $\lim_{x\to c} f(x) = L$.

SOLUTION: $\lim_{x\to c} f(x) = L$ iff $\forall\ \epsilon > 0\ \exists\ \delta > 0 : 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

worth 2 pts. -1 for any mistake (up to 2 mistakes).

(b) Use the definition in (a) to prove that $\lim_{x\to 2}(4x-3) = 5$.

SOLUTION: Given $\epsilon > 0$, let $\delta \le \frac{\epsilon}{4}$.

Then $0 < |x-2| < \delta \Rightarrow |(4x-3)-5| = |4x-8| = 4|x-2| < 4\cdot\frac{\epsilon}{4} = \epsilon$.

Worth 3 pts: Choice of δ: 1 pt.

Reduction of $|(4x-3)-8|$: 1 pt.

$< \epsilon$: 1 pt.

15. (4 points) A certain radioactive substance loses 20% of its mass in one year. What is the half-life of the radioactive substance?

SOLUTION: $Q = Q_0 e^{-kt}$; $t = 1$ yr $\Rightarrow Q = .8Q_0 : .8Q_0 = Q_0 e^{-k} \Rightarrow$

$$.8 = e^{-k} \Rightarrow \frac{10}{8} = e^k \Rightarrow k = \ln\left(\frac{10}{8}\right) = \ln\left(\frac{5}{4}\right).$$

$$kh = \ln 2 \Rightarrow h = \frac{\ln 2}{k} = \frac{\ln 2}{\ln\left(\frac{5}{4}\right)}.$$

decay formula: 1 pt.	computation of k: 1 pt.
relationship of h to k: 1 pt.	finding h: 1 pt.

16. (4 points) A certain commodity grows in value V at the rate of $\frac{dV}{dt} = 30e^{0.03t}$ at time t, $0 \le t \le 10$. If its initial value is \$50, what is its value at time $t = 10$? [Hint: Use the Fundamental Theorem; you need not simplify.]

SOLUTION:
$$\int_0^{10} \frac{dV}{dt}\,dt = V(10) - V(0) \Rightarrow V(10) = V(0) + \int_0^{10} 30e^{0.03t}dt$$

Fund. theorem or equivalent: 1 pt.	$= 50 + 30\left(\frac{e^{.03t}}{.03}\Big|_0^{10}\right)$
correct antiderivative: 1pt.	$= 50 + \frac{30}{.03}\left(e^{.3} - 1\right)$
correct initial value: 1 pt.	$= 50 + 1000(e^{.3} - 1)$
correct answer: 1 pt.	

17. (3 points) Estimates of $\int_a^b f(x)dx$ by the trapezoidal rule with $n = 5$ and $n = 10$ are 1.057 and 1.125, respectively. Determine the concavity of f, given that the concavity does not change over (a, b).

SOLUTION: $T_{10} > T_5 \Rightarrow f$ is concave downward.

Observ.: $T_{10} > T_5$: 1 pt.

indication that T_n is linear estimate: 1 pt.

Correct answer: 1pt.

18. (6 points) Consider the function

$$f(x) = \begin{cases} e^x - 1, & x \leq 0 \\ \ln(1+x), & x > 0. \end{cases}$$

(a) Is f continuous at 0? Justify your answer.

SOLUTION:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (e^x - 1) = 0 \qquad \text{(1 pt.)}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln(1+x) = 0 \qquad \text{(1 pt.)}$$

$$\therefore \lim_{x\to 0} f(x) = 0 = f(0) \qquad \text{(1 pt).}$$

(b) Is f differentiable at 0? Justify your answer.

SOLUTION:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

$$h < 0: \lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{\ln(1+h)}{h} = \lim_{h\to 0^-} \frac{\frac{1}{1+h}}{1} = 1 \qquad \text{(1 pt.)}$$

$$h > 0: \lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{e^h - 1}{h} = \lim_{h\to 0^+} \frac{e^h}{1} = 1 \qquad \text{(1 pt.)}$$

$$\therefore f'(0) = 1. \qquad \text{(1 pt.)}$$

19. (4 points) Given $x^2 - xy + y^2 = 5$, calculate $\frac{dy}{dx}$.

SOLUTION:

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \qquad \text{(2 pts.)}$$

$$(2y - x)\frac{dy}{dx} = y - 2x \qquad \text{(1 pt.)}$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x} \qquad \text{(1 pt.)}$$

20. (6 points) The function $I(t) = \dfrac{200t}{t^3+1}$ represents the rate of infection of a disease (the rate at which a disease spreads), in people per month, where t is the time, in months, since the disease broke out.

(a) At what time is the rate of infection the highest?

SOLUTION: $I'(t) = \dfrac{200(t^3+1) - 200t(3t^2)}{(t^3+1)^2} = \dfrac{200t^3 + 200 - 600t^3}{(t^3+1)^2}$

$= \dfrac{-400t^3 + 200}{(t^3+1)^2} = 0 \Rightarrow 400t^3 = 200 \Rightarrow t^3 = \dfrac{1}{2} \Rightarrow t = \dfrac{1}{\sqrt[3]{2}}.$

$I(0) = 0$; $I\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$. Unique cr. pt. $\Rightarrow$ max.

Compute $I'(t)$: 1 pt.
Find crit. pt.: 1 pt.
Argue for max: 1 pt.

(b) Suppose $N(t) = \displaystyle\int_0^t I(x)dx$. Explain in the context of the problem what the function $N(t)$ measures.

SOLUTION: Since $I(t)$ is a rate of change, $N(t)$ is the total change, or the total number of people infected by time t. (2 pts.)

(c) Calculate $\dfrac{dN}{dt}$.

SOLUTION: $\dfrac{dN}{dt} = I(t)$. (1 pt.)

21. (6 points) Given the function $F(x) = \int_0^x \sqrt{t^2+4}\, dt$

(a) Explain why F is increasing on $(0, \infty)$.

SOLUTION: $F'(x) = \sqrt{x^2+4} > 0 \Rightarrow F$ is increasing.

(b) Explain why there is a point p between 0 and 1 such that $F(p) = 2$.

SOLUTION: $F(1) = \displaystyle\int_0^1 \sqrt{t^2+4}\, dt > \underbrace{\int_0^1 2dt}_{=2}$ (Since $\sqrt{t^2+4} > \sqrt{4}$ on $[0,1]$.)

$F(0) = \displaystyle\int_0^0 \sqrt{t^2+4}\, dt = 0 < 2$. Since F is continuous, there must be some $p \in (0,1)$ such that $F(p) = 2$, by IVT.

$F(0) < 2$: 1 pt; $F(1) > 2$: 1 pt; IVT: 1 pt.

22. (9 points) Suppose g is a differentiable function defined on all of $\mathbb{R}$ and having the properties

(1) For all real numbers u and v, $g(u+v) = g(u) + g(v) + 2uv$, and

(2) $\lim_{h\to 0} \frac{g(h)}{h} = 3$.

(a) Use the definition of derivative to find $g'(x)$.

SOLUTION:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{g(x)+g(h)+2xh-g(x)}{h} = \lim_{h\to 0} \frac{g(h)}{h} + 2x = \text{2x + 3}.$$

def. of der. 1 pt. Using (1): 1 pt. Using (2): 1 pt. Finding limit: 1 pt.

(b) Use Property (1) to find $g(0)$.

SOLUTION:

$g(0) = g(0+0) = g(0) + g(0) + 0 \Rightarrow g(0) = 2g(0) \Rightarrow g(0) = 0$.

Using (1): 1 pt.

Setting u or $v = 0$: 1 pt.

Answer: 1 pt.

(c) Find $g(x)$. Justify your answer.

SOLUTION: $g'(x) = 2x + 3 \Rightarrow g(x) = x^2 + 3x + c$

$$0 = g(0) = c \Rightarrow g(x) = x^2 + 3x$$

antiderivative: 1 pt.
$c = 0$: 1 pt.